Vraag 1

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n-3^n}{4^n+2^n}$ $\qquad a_n = \dfrac{4^n-3^n}{4^n+2^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^n-3^n}{4^n+2^n} =$$

$$= \lim_{n\to\infty} \frac{4^n\left(1-\left(\frac{3}{4}\right)^n\right)}{4^n\left(1+\left(\frac{1}{2}\right)^n\right)} = \lim_{n\to\infty} \frac{1-\left(\frac{3}{4}\right)^n}{1+\left(\frac{1}{2}\right)^n} = 1 \neq 0$$

$\Rightarrow$ divergent

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$ $\qquad$ integraaltest: $\displaystyle\int_1^{\infty} \frac{x}{e^x}\,dx = \int_1^{\infty} x e^{-x}\,dx$

partiëel integreren

$$= -x e^{-x}\Big|_1^{\infty} - \int_1^{\infty} -e^{-x}\,dx = 0 \underset{(--)}{+} e^{-1} + \int_1^{\infty} e^{-x}\,dx$$

$$= -x e^{-x}$$

$$= e^{-1} - e^{-x}\Big|_1^{\infty} = e^{-1} - (0 - e^{-1})$$

reeks ook convergent $\Longleftarrow$ $= \dfrac{2}{e} < \infty$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{100^n}{n^{200}}$ $\qquad$ ratiotest: $\left|\left(\dfrac{100^{n+1}}{(n+1)^{200}}\right) \Big/ \left(\dfrac{100^n}{n^{200}}\right)\right| =$

$$= \left|\frac{100^{n+1}}{(n+1)^{200}} \cdot \frac{n^{200}}{100^n}\right| = 100 \cdot \frac{n^{200}}{(n+1)^{200}} \xrightarrow[n\to\infty]{} 100 > 1$$

dezelfde limietwaarde boven- en onder de streep

$\Rightarrow$ divergent

$\boxed{\text{Vraag 2}}$

$$E(T) = \cos(T^2) - 1 + \frac{1}{2}T^4$$
$$\rho(T) = T^2(T - \sin(T))^2$$

(a) $\cos(T) = 1 - \frac{1}{2}T^2 + \frac{1}{24}T^4 + \ldots$

$\sin(T) = T - \frac{1}{6}T^3 + \frac{1}{120}T^5 - \frac{1}{720}T^7 + \ldots$

$\Rightarrow \cos(T^2) = 1 - \frac{1}{2}T^4 + \frac{1}{24}T^8 + \ldots$

$\cos(T^2) - 1 + \frac{1}{2}T^4 = \frac{1}{24}T^8 + \ldots = E(T)$ t/m T^8

en $(T - \sin(T))^2 = \left(\frac{1}{6}T^3 - \frac{1}{120}T^7 + \frac{1}{720}T^9 \ldots\right)^2$

$= \frac{1}{36}T^6 + \ldots$

$T^2(T - \sin(T))^2 = \frac{1}{36}T^8 + \ldots = \rho(T)$ t/m T^8

(b) $$\lim_{T \to 0} \frac{E(T)}{\rho(T)} = \lim_{T \to 0} \frac{\frac{1}{24}T^8 + \ldots}{\frac{1}{36}T^8 + \ldots} = \frac{\frac{1}{24}}{\frac{1}{36}} = \frac{36}{24} = \frac{3}{2}$$

$\boxed{\text{Vraag 3}}$

$$\begin{cases} y' = 4x - 2xy \\ y(0) = 5 \end{cases}$$

Scheiding van variabelen

$$y' = 2x(2-y)$$

$$\Leftrightarrow \frac{dy}{dx} = 2x(2-y)$$

$$\Leftrightarrow \int \frac{dy}{2-y} = \int 2x\,dx$$

$$\Leftrightarrow -\ln(2-y) = x^2 + c$$

$$\Leftrightarrow 2-y = e^{-x^2} \cdot \tilde{c}$$

$$\Leftrightarrow -y = \tilde{c}e^{-x^2} - 2$$

$$\Leftrightarrow y = 2 - \tilde{c}e^{-x^2}$$

$$y(0) = 2 - \tilde{c}.e^0 = 2 - \tilde{c} = 5 \Rightarrow \tilde{c} = -3$$

$$\Downarrow$$

$$\boxed{y(x) = 2 + 3e^{-x^2}}$$

Vraag 4

$$\begin{cases} a' = -a + 2b & , a(0) = 1 \\ b' = a - 2b & , b(0) = 5 \end{cases}$$

matrix $\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$ eigenwaarden: $\begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow 2 + \lambda + 2\lambda + \lambda^2 - 2 = 0$$
$$\Leftrightarrow \lambda^2 + 3\lambda = 0$$
$$\Leftrightarrow \lambda(\lambda+3) = 0 \Leftrightarrow \lambda = 0 \text{ of } \lambda = -3$$

- eigenvectoren:

bij $\lambda = 0$: $-x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2$
$\Rightarrow \alpha \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\alpha \neq 0$

bij $\lambda = 3$: $2x_1 + 2x_2 = 0 \Rightarrow x_1 = -x_2$
$\Rightarrow \beta \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\beta \neq 0$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{0.t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

$$\Rightarrow \begin{cases} a(t) = 2c_1 + c_2 e^{-3t} \\ b(t) = c_1 - c_2 e^{-3t} \end{cases}$$

$$\begin{cases} a(0) = 1 \\ b(0) = 5 \end{cases} \Rightarrow \begin{cases} 1 = 2c_1 + c_2 \\ 5 = c_1 - c_2 \end{cases} \overset{\text{oplossen}}{\Longrightarrow} \begin{cases} c_1 = 2 \\ c_2 = -3 \end{cases}$$

$$\Longrightarrow \begin{cases} a(t) = 4 - 3e^{-3t} \\ b(t) = 2 + 3e^{-3t} \end{cases}$$

$t \to \infty$: $a(t) \to 4 = [A](t=\infty)$ en $b(t) \to 2 = [B](t=\infty)$

$\boxed{\text{Vraag 5}}$

$$\mathcal{E}(x,y) = \frac{x+y}{x-y} + 1200 \qquad (x \neq y)$$

(a)

$$\frac{\partial \mathcal{E}}{\partial x} = \frac{1.(x-y) - 1.(x+y)}{(x-y)^2} = \frac{x-y-x-y}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

quotiëntregel

$$\frac{\partial \mathcal{E}}{\partial y} = \frac{1.(x-y) \underset{(--)}{+} 1.(x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\Rightarrow \text{grad}(\mathcal{E}) = \nabla \mathcal{E} = \begin{pmatrix} \dfrac{-2y}{(x-y)^2} \\ \dfrac{2x}{(x-y)^2} \end{pmatrix}$$

(b)

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} = \frac{0.(x-y)^2 + 2y(x-y).2}{(x-y)^4} = \frac{4xy - 4y^2}{(x-y)^4}$$

$$\frac{\partial^2 \mathcal{E}}{\partial y^2} = \frac{0.(x-y)^2 - 2x(x-y).-2}{(x-y)^4} = \frac{-4xy + 4x^2}{(x-y)^4}$$

$$\frac{\partial^2 \mathcal{E}}{\partial x \partial y} = \frac{\partial^2 \mathcal{E}}{\partial y \partial x} = \frac{2(x-y)^2 - 2x(x-y).2}{(x-y)^4} = \frac{2x^2 \cancel{-4xy} + 2y^2 - 4x^2 \cancel{+4xy}}{(x-y)^4}$$

$$\Rightarrow \mathcal{H} = \begin{pmatrix} \dfrac{\partial^2 \mathcal{E}}{\partial x^2} & \dfrac{\partial^2 \mathcal{E}}{\partial y \partial x} \\ \dfrac{\partial^2 \mathcal{E}}{\partial x \partial y} & \dfrac{\partial^2 \mathcal{E}}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \dfrac{4xy-4y^2}{(x-y)^4} & \dfrac{-2x^2+2y^2}{(x-y)^4} \\ \dfrac{-2x^2+2y^2}{(x-y)^4} & \dfrac{-4xy+4x^2}{(x-y)^4} \end{pmatrix}$$

$$= \frac{-2x^2+2y^2}{(x-y)^4}$$

en $\nabla \cdot (\nabla \mathcal{E})$

$$= \Delta \mathcal{E} = \frac{\partial^2 \mathcal{E}}{\partial x^2} + \frac{\partial^2 \mathcal{E}}{\partial y^2} = \frac{4xy - 4y^2}{(x-y)^4} + \frac{-4xy+4x^2}{(x-y)^4} = \frac{4(x^2-y^2)}{(x-y)^4}$$

(c) $\vec{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \Rightarrow \|b\| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$$\text{grad}\, \mathcal{E} \text{ in } (2,1): \begin{pmatrix} \frac{-2.1}{(2-1)^2} \\ \frac{2.2}{(2-1)^2} \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$D_{\vec{b}} \mathcal{E}\Big|_p = \vec{b} \cdot \text{grad}\, \mathcal{E} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{10}} (-2 - 12)$$

$$= \frac{-14}{\sqrt{10}}$$

(d) $$\nabla \times (\nabla \mathcal{E}) = \nabla \times \begin{pmatrix} \frac{\partial \mathcal{E}}{\partial x} \\ \frac{\partial \mathcal{E}}{\partial y} \\ \frac{\partial \mathcal{E}}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{E}}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{E}}{\partial y}\right) \\ \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{E}}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{E}}{\partial z}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{E}}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{E}}{\partial x}\right) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \mathcal{E}}{\partial y \partial z} - \frac{\partial^2 \mathcal{E}}{\partial z \partial y} \\ \frac{\partial^2 \mathcal{E}}{\partial z \partial x} - \frac{\partial^2 \mathcal{E}}{\partial x \partial z} \\ \frac{\partial^2 \mathcal{E}}{\partial x \partial y} - \frac{\partial^2 \mathcal{E}}{\partial y \partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \vec{0}$$

$\boxed{\text{Vraag 6}}$

$$\begin{cases} y' = x^2 y \\ y(0) = 162 \end{cases}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$-x^2 y = -\sum_{n=0}^{\infty} a_n x^{n+2} = -\sum_{n=2}^{\infty} a_{n-2} x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow a_1 + 2a_2 x + \sum_{n=2}^{\infty} \left[(n+1)a_{n+1} - a_{n-2}\right] x^n = 0$$

$$\Rightarrow \boxed{a_1 = a_2 = 0} \text{ en } a_{n+1} = \frac{1}{n+1} a_{n-2} \quad n = 2,3,4,\ldots$$

$$\boxed{a_0 = y(0) = 162}$$

$a_4 = 0, a_7 = 0$, etc ...

$a_5 = 0, a_8 = 0$, etc ..

$$a_3 = \frac{162}{3} = 54, \quad a_6 = \frac{54}{6} = 9, \quad a_9 = \frac{9}{9} = 1$$

$$\Rightarrow \boxed{a_0 = 162, a_1 = 0, a_2 = 0, a_3 = 54, a_4 = 0, a_5 = 0, a_6 = 9, a_7 = 0, a_8 = 0, a_9 = 1}$$

$$y(x) = 162 + 54x^3 + 9x^6 + x^9 + \cdots \quad (\equiv 162 e^{\frac{1}{3}x^3})$$

$\boxed{\text{Vraag 7}}$

Brusselator

$$\begin{cases} x' = 2-5x+x^2y \\ y' = 4x-x^2y \end{cases}$$

(a) kritieke punt: $\begin{cases} 2-5x+x^2y=0 \\ 4x-x^2y=0 \end{cases}$

$\Leftrightarrow \begin{cases} 2-5x+x^2y=0 \\ x(4-xy)=0 \end{cases} \longrightarrow x=0$ of $4-xy=0$

1e vgl

"$2=0$" ↯

kan niet

dus

$y=\frac{4}{x} \xrightarrow{\text{1e vgl}} 2-5x+x^2\cdot\frac{4}{x}=0 \Rightarrow 2-5x+4x=0$

$\Rightarrow x=2 \Rightarrow y=\frac{4}{2}=2$

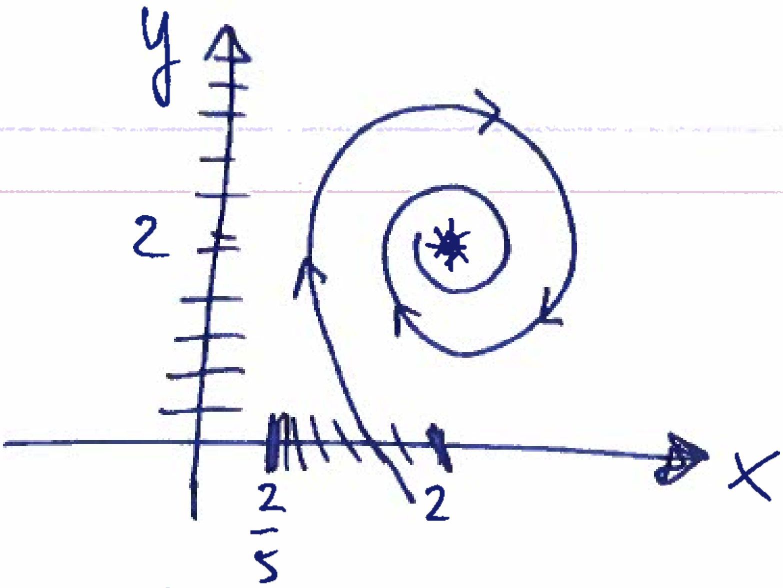
$\boxed{(2,2)}$

(b) Jacobiaan $J=\begin{pmatrix} -5+2xy & x^2 \\ 4-2xy & -x^2 \end{pmatrix}$

in $x=2, y=2$: $J=\begin{pmatrix} -5+8 & 2^2 \\ 4-8 & -2^2 \end{pmatrix}=\begin{pmatrix} 3 & 4 \\ -4 & -4 \end{pmatrix}$

$$\left.\begin{array}{l} \text{spoor} = 3-4=-1<0 \\ \det = -12+16=4>0 \\ D = sp^2-4.\det = 1-16=-15<0 \end{array}\right\} \Rightarrow \text{stabiele spiraal}$$

(c)



$$x=0: \quad \frac{dy}{dx} = \frac{4x - x^2 y}{2 - 5x + x^2 y} = \frac{0}{2} = 0$$

$$y=0: \quad \frac{dy}{dx} = \frac{4x}{2-5x}$$

$\boxed{\text{Vraag 8}}$

$$\begin{cases} u_t - 7u_x = -3u \\ u(x,0) = \sin(\pi x) \end{cases}$$

$$u(x,t) = e^{-3t} \sin(\pi(x+7t))$$

$$u(x,0) = e^{0} \sin(\pi(x+0)) = \sin(\pi x) \checkmark$$

$$u_t = -3e^{-3t} \sin(\pi(x+7t)) + e^{-3t} \cdot 7\pi \cos(\pi(x+7t))$$

$$u_x = e^{-3t} \pi \cos(\pi(x+7t))$$

$$\Rightarrow u_t - 7u_x = -3e^{-3t} \sin(\pi(x+7t))$$
$$+ \cancel{7\pi e^{-3t} \cos}(\pi(x+7t))$$
$$- \cancel{7\pi e^{-3t} \cos}(\pi(x+7t))$$
$$= -3e^{-3t} \sin(\pi(x+7t))$$
$$= -3u$$