$\boxed{\text{Vraag 1}}$

(a) $$\sum_{n=2}^{\infty} \frac{1}{n^2+n}$$

comparison test : $\frac{1}{n^2+n} < \frac{1}{n^2}$ en $\sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$

(b) $$\sum_{n=1}^{\infty} n^{-1/6}$$

integral test : $\int_1^{\infty} x^{-1/6}\, dx = \frac{6}{7} x^{7/6} \Big|_{x=1}^{x=\infty} = \infty$

(c) $$\sum_{n=0}^{\infty} \frac{2n-1}{2^n}$$

ratiotest : $\left(\frac{2(n+1)-1}{2^{n+1}}\right) \Big/ \left(\frac{2n-1}{2^n}\right) = \frac{2n+1}{2^{n+1}} \cdot \frac{2^n}{2n-1}$

$$= \frac{2+\frac{1}{n}}{2-\frac{1}{n}} \cdot \frac{1}{2} \underset{n\to\infty}{\to} \frac{1}{2} < 1$$

(d) $$\sum_{n=0}^{\infty} n^{\frac{4}{3}}$$

$$\lim_{n\to\infty} n^{\frac{4}{3}} \neq 0$$

$\boxed{\text{Vraag 2}}$

$$\begin{cases} y' = -\dfrac{y^2}{x} \\ y(1) = \dfrac{1}{3} \end{cases}$$

scheiden van variabelen

$$\frac{dy}{dx} = -\frac{y^2}{x} \Leftrightarrow \int -\frac{dy}{y^2} = \int \frac{dx}{x}$$

$$\Leftrightarrow \quad \frac{1}{y} = \ln(x) + c$$

$$\Leftrightarrow \quad y = \frac{1}{\ln(x) + c}$$

$$y(1) = \frac{1}{\ln(1) + c} = \frac{1}{c} = \frac{1}{3} \Rightarrow c = 3$$

$$\Rightarrow \quad y(x) = \frac{1}{\ln(x) + 3}$$

Vraag 3

$$T(x) = \cos(x^2) - e^{x^4}$$

$$\mathcal{E}(x) = \sin(x^4)$$

(a) $$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \ldots$$

$$= 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 - \frac{1}{6!}x^{12} + \ldots$$

$$e^{x^4} = 1 + (x^4) + \frac{(x^4)^2}{2!} + \frac{(x^4)^3}{3!} + \ldots$$

$$= 1 + x^4 + \frac{1}{2}x^8 + \frac{1}{6}x^{12} + \ldots$$

$$\mathcal{E}(x) = \sin(x^4) = (x^4) - \frac{(x^4)^3}{3!} + \frac{(x^4)^5}{5!} + \ldots$$

$$= x^4 - \frac{1}{6}x^{12} - \ldots$$

$$\Rightarrow T(x) = \left(\underline{1} - \frac{1}{2}x^4 + \frac{1}{24}x^8 - \frac{1}{6!}x^{12} \ldots\right) - \left(\underline{1} + x^4 + \frac{1}{2}x^8 + \frac{1}{6}x^{12} \ldots\right)$$

$$= -\frac{3}{2}x^4 - \frac{11}{24}x^8 - \left(\frac{1}{6!} + \frac{1}{6}\right)x^{12} \ldots \quad = 121/720$$

(b) $$\lim_{x\to 0} \frac{T(x)}{\mathcal{E}(x)} = \lim_{x\to 0} \frac{-\frac{3}{2}x^4 - \frac{11}{24}x^8 - \left(\frac{1}{6!} + \frac{1}{6}\right)x^{12} \ldots}{x^4 - \frac{1}{6}x^{12} \ldots}$$

$$\frac{121}{720}$$

$$= \lim_{x\to 0} \frac{-\frac{3}{2} - \frac{11}{24}x^4 - \left(\frac{1}{6!} + \frac{1}{6}\right)x^8 -}{1 - \frac{1}{6}x^8 \ldots}$$

$$\frac{121}{720}$$

$$= -\frac{3}{2}$$

$\boxed{\text{Vraag 4}}$ $f(x,y) = (x-4)^2 + 2(y+5)^2 + 2020$

(a) gradiënt: $\text{grad}(f) = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x-8 \\ 4y+20 \end{pmatrix}$

(b) $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = 0$, $\frac{\partial^2 f}{\partial y^2} = 4$

$\Rightarrow \mathcal{H} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ en $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6$

(c) $D_{\vec{b}} f \big|_p = D_{(3,4)^T} f \big|_{(6,1)}$

$= \text{grad}(f)\big|_{(6,1)} \bullet \frac{(3,4)^T}{\|(3,4)^T\|}$ $\sqrt{3^2+4^2} = 5$

$= \begin{pmatrix} 4 \\ 24 \end{pmatrix} \bullet \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \frac{12}{5} + \frac{96}{5} = \frac{108}{5}$

(d) $\det \mathcal{H} = 8 > 0$ en $\Delta f = 6 > 0$

stationaire punten: $\begin{cases} 2x-8=0 \\ 4y+20=0 \end{cases} \Rightarrow (x,y) = (4,-5)$ $\Big\}$ minimum

$\boxed{\text{Vraag 5}}$

$$\begin{cases} y' = y + x^2 \\ y(0) = -2 \end{cases}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y(0) = a_0 + a_1 0 + a_2 0^2 + \dots = -2$$

$$\Rightarrow \boxed{a_0 = -2}$$

$$\Downarrow$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

invullen in DV: $\displaystyle\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n + x^2$

uitschrijven: $a_1 + 2a_2 x + 3a_3 x^2 + \dots = \underbrace{a_0}_{=-2} + a_1 x + \dots + x^2$

$$\Rightarrow \begin{aligned} & a_1 = a_0 = -2 \\ & 2a_2 = a_1 \Rightarrow a_2 = \tfrac{1}{2} a_1 = -1 \\ & 3a_3 = \underbrace{a_2 + 1}_{=0} \Rightarrow a_3 = 0 \\ & 4a_4 = a_3 \quad \Rightarrow a_4 = 0 \\ & 5a_5 = a_4 \quad \Rightarrow a_5 = 0 \\ & \text{etcetera} \quad \text{etcetera} \end{aligned}$$

$$\Longrightarrow y(x) = -2 - 2x - x^2$$

Vraag 6

$$\begin{cases} x' = y - 2xy \\ y' = 2xy - x \end{cases}$$

(a) kritieke punten: $\begin{cases} y(1-2x) = 0 \\ x(2y-1) = 0 \end{cases}$

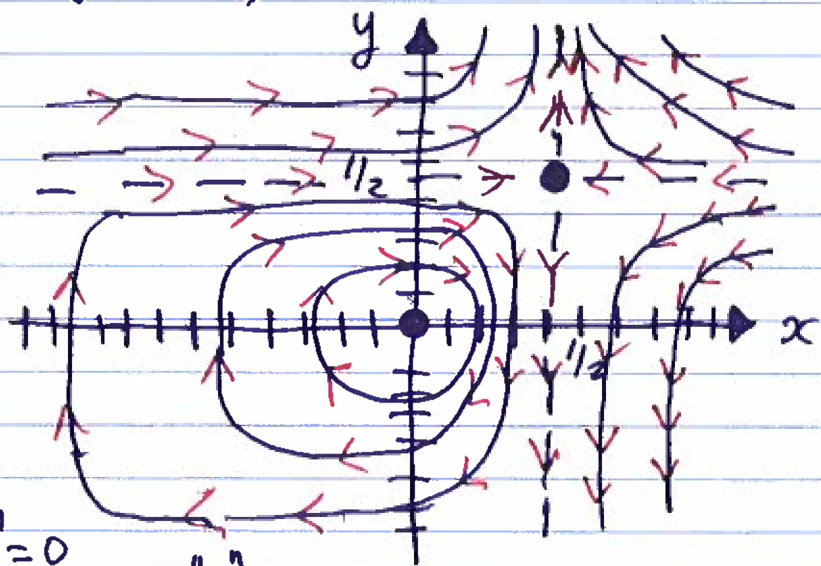
$\Rightarrow (0,0)$ en $(\frac{1}{2}, \frac{1}{2})$

(b) Jacobiaan: $\begin{pmatrix} -2y & 1-2x \\ 2y-1 & 2x \end{pmatrix}$

in $(0,0)$: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\Rightarrow$ centrumpunt $\begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$

in $(\frac{1}{2}, \frac{1}{2})$: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ zadelpunt $\begin{cases} \lambda_1 = -1 \\ \lambda_2 = 1 \end{cases}$

(c) $\dfrac{dy}{dx} = \dfrac{x(2y-1)}{y(1-2x)} = \begin{cases} \to \infty \text{ als } y=0 \text{ of } x = \frac{1}{2} \\ \to 0 \text{ als } x=0 \text{ of } y = \frac{1}{2} \end{cases}$



op $\begin{cases} x = -1 \\ y = 0 \end{cases}$ $\to$ $\begin{cases} x' = 0 \\ y' = 1 > 0 \end{cases}$ $\Rightarrow$ "$\uparrow$"

$\boxed{\text{Vraag 7}}$

$$\begin{cases} u_t = u u_{xx} \\ u(0,t) = u(1,t) = \dfrac{1}{1-t} \quad (0<t<1) \end{cases}$$

aanname: $u(x,t) = \dfrac{X(x)}{1-t}$

$$\Rightarrow \quad u_t = X \cdot \frac{1}{(1-t)^2} = \frac{X}{1-t} \cdot \frac{X''}{1-t} = u u_{xx} \quad (t \neq 1)$$

$X \neq 0$ ($\Rightarrow$ niet-triviale oplossingen)

$$\Rightarrow X'' = 1 \Rightarrow X' = x + c$$

$$X(x) = \tfrac{1}{2}x^2 + cx + d$$

$$u(0,t) = \left.\frac{\frac{1}{2}x^2 + cx + d}{1-t}\right|_{x=0} = \frac{d}{1-t} = \frac{1}{1-t} \Rightarrow d = 1$$

$$\Rightarrow u(x,t) = \frac{\frac{1}{2}x^2 + cx + 1}{1-t}$$

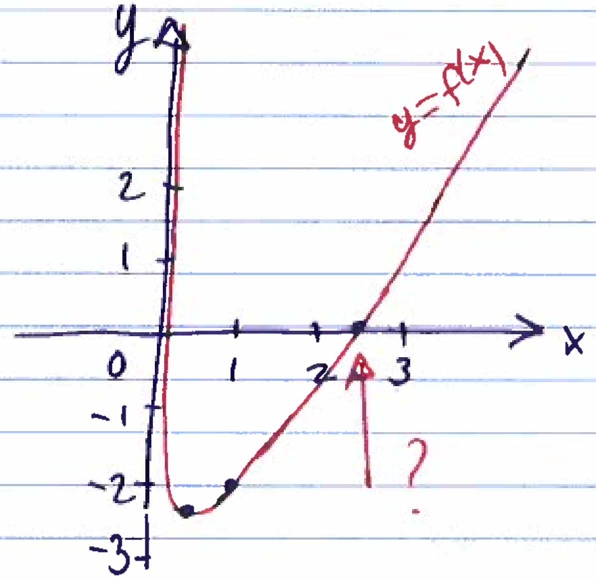
$$u(1,t) = \frac{\frac{1}{2} + c + 1}{1-t} = \frac{1}{1-t} \Rightarrow c = -\frac{1}{2}$$

$$\Rightarrow X(x) = \tfrac{1}{2}x^2 - \tfrac{1}{2}x + 1$$

$$\Longrightarrow u(x,t) = \frac{\frac{1}{2}x^2 - \frac{1}{2}x + 1}{1-t}$$

$\boxed{\text{Vraag 8}}$

$$f(x) = 2x - \ln(x) - 4 \quad , \; x > 0$$



$f(x) = 0$

NR :

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad , \; i = 0, 1, 2, \ldots$$

$$\begin{cases} f(x) = 2x - \ln(x) - 4 \\ f'(x) = 2 - \frac{1}{x} \end{cases}$$

$x_0 = 3$

$$x_1 = x_0 - \frac{2x_0 - \ln(x_0) - 4}{2 - \frac{1}{x_0}} = 2{,}4592$$

$$x_2 = x_1 - \frac{2x_1 - \ln(x_1) - 4}{2 - \frac{1}{x_1}} = 2{,}4475 \quad (2{,}447549)$$

$$x_3 = x_2 - \frac{2x_2 - \ln(x_2) - 4}{2 - \frac{1}{x_2}} = 2{,}4475 \quad (2{,}447542)$$

$$x_4 = x_3 - \frac{2x_3 - \ln(x_3) - 4}{2 - \frac{1}{x_3}} = 2{,}4475 \quad (2{,}447542)$$

etcetera

$\boxed{\text{Vraag 9}}$ $\begin{cases} 2y'' - y' - y = 4xe^{2x} \\ y(0) = -\frac{28}{25},\ y'(0) = \frac{39}{25} \end{cases}$

algemene homogene oplossing:

$2y_h'' - y_h' - y_h = 0 \Rightarrow 2\lambda^2 - \lambda - 1 = 0$

$y_h = e^{\lambda x}$ $\Rightarrow \lambda_1 = 1, \lambda_2 = -\frac{1}{2}$

$\Rightarrow y_h = c_1 e^{x} + c_2 e^{-\frac{1}{2}x}$

een particuliere oplossing:

laat $y_p = (Ax+B)e^{2x} \Rightarrow y_p' = Ae^{2x} + 2(Ax+B)e^{2x}$

$= (A+2B+2Ax)e^{2x}$

$\Rightarrow y_p'' = 2Ae^{2x} + 2(A+2B+2Ax)e^{2x}$

$= e^{2x}(2A+2A+4B+4Ax) = e^{2x}(4A+4B+4Ax)$

invullen in DV: $2e^{2x}(4A+4B+4Ax)$

$- e^{2x}(A+2B+2Ax)$

$- e^{2x}(Ax+B) = 4xe^{2x}$

delen door $e^{2x} \neq 0$: $8A + 8B + 8Ax - A - 2B - 2Ax - Ax - B = 4x$

$\Leftrightarrow 7A + 5B + 5Ax = 4x$

$\Rightarrow \begin{cases} 5A = 4 & \Rightarrow A = 4/5 \\ 7A + 5B = 0 & \Leftrightarrow B = -\frac{28}{25} \end{cases} \Rightarrow y_p = \left(\frac{4}{5}x - \frac{28}{25}\right)e^{2x}$

de volledige oplossing: (gebruik de twee beginvoorwaarden om c_1 en c_2 te bepalen)

$y = c_1 e^{x} + c_2 e^{-\frac{1}{2}x} + \left(\frac{4}{5}x - \frac{28}{25}\right)e^{2x}$

$y(0) = c_1 + c_2 - \frac{28}{25} = -\frac{28}{25} \Rightarrow c_1 + c_2 = 0$

$y' = c_1 e^{x} - \frac{1}{2}c_2 e^{-\frac{1}{2}x} + \frac{4}{5}e^{2x} + 2\left(\frac{4}{5}x - \frac{28}{25}\right)e^{2x}$

$y'(0) = c_1 - \frac{1}{2}c_2 + \frac{4}{5} - \frac{56}{25} = \frac{39}{25} \Rightarrow c_1 - \frac{1}{2}c_2 = \frac{75}{25} = 3$

$\Rightarrow c_1 = 2, c_2 = -2$ $\Rightarrow \boxed{y = 2e^{x} - 2e^{-\frac{1}{2}x} + \left(\frac{4}{5}x - \frac{28}{25}\right)e^{2x}}$