Problem 1.

(**Question 1A**)

Using the trigonometric relations given on the previous page prove that

$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$.

Solution:

One can prove this using different ways. For me the easiest way is to use $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta \mp \sin\alpha\sin\beta$. Then

$\cos(\alpha+\beta)+\cos(\alpha-\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta = 2\cos\alpha\cos\beta.$ Or,

$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$.

(**Question 1B**)

A standing wave is written using real functions as

$A_1(x,t)=A_0\cos(\omega t)\cos(kx-\pi/2)$

Calculate the position of nodes in this standing wave. What is the distance between two neighbouring nodes?

Solution:

Nodes are @ $\cos(kx-\pi/2)=0$, i.e. $kx-\pi/2=\pi/2+\pi n$, where n is any integer. Thus, $x=(\pi+\pi n)/k$. Since n is *any* integer, you can also write $x=\pi n/k$.

Taking into account $k=2\pi/\lambda$, you can also write the answer as $x=\lambda n/2$.

The distance between nodes is $\lambda/2$.

(**Question 1C**)

Now, to the standing wave $A_1(x,t)$ from Question 1B, we add a travelling wave

$A_2(x,t)=\frac{1}{2}A_0\cos(\omega t+kx+\pi/2)$,

Show that the result of their interference $A_1(x,t)+A_2(x,t)$ is another travelling wave.

In what direction this wave propagates? What is its phase velocity?

Solution:

Using the formula from Question 1A, you can realise that $A_1(x,t)$ can be written as $A_1(x,t)=A_0\cos(\omega t)\cos(kx-\pi/2)=(1/2)A_0\cos(\omega t+kx-\pi/2)+(1/2)A_0\cos(\omega t-kx+\pi/2)$ a sum of two travelling waves. The first term is a wave travelling in the negative x direction, so as the A_2 wave. Using the fact that $\cos(\alpha+\pi)=-\cos(\alpha)$, you can see that

$(1/2)A_0\cos(\omega t+kx-\pi/2)+(1/2)A_0\cos(\omega t+kx+\pi/2)$

$=(1/2)A_0\cos(\omega t+kx-\pi/2)+(1/2)A_0\cos(\omega t+kx-\pi/2+\pi)$

$=(1/2)A_0\cos(\omega t+kx-\pi/2)-(1/2)A_0\cos(\omega t+kx-\pi/2)=0$

So, the first term in the A_1 wave and the A_2 wave interfere destructively. The only remaining part is the second term in the A_1 wave: $A_1(x,t)+A_2(x,t)=(1/2)A_0\cos(\omega t-kx+\pi/2)$. This is a wave travelling in the ***positive*** x direction.

Problem 2.

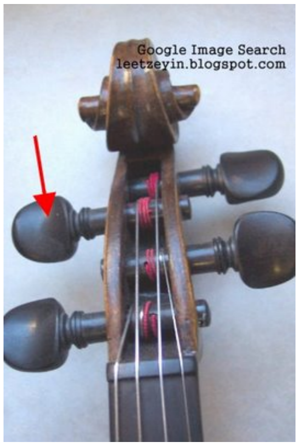
(**Question 2A**)

Write down the formula for the lowest frequency ω (main tone) of a string with length L and the phase velocity of the string wave v.

Solution:

Using $v = \omega / k = \omega\lambda / (2\pi)$ and the condition for the main tone $L = \lambda / 2$, one can easily get $\omega = \pi v / L$.

(**Question 2B**)



A violin (=viool in Dutch) is tuned by turning the pegs (see the picture on the left).

Is it the length of the string *L* or the phase velocity *v* of the waves propagating along the string, which change during tuning?

On the other hand, while playing, the musician presses the string against the deck at different positions to produce different notes (see the photo on the right).

Explain why the tone depends on where the string is pressed.

<u>Solution</u>:
When a violin is tuned, one hardly changes the length of the string. And one certainly does not change the length of the part which can vibrate. What is really tuned is the tension (spanning) of the string. This changes the velocity *v* of the waves travelling along the string.
When a musician plays the violin, she/he fixes the string at different positions, thus reducing the length of the part of the string *L*, which can vibrate. This increases the frequency so that one can one make different notes.

(**Question 2C**)
In an orchestra one uses a number of stringed instruments of different size.

Why one needs such a diversity of instruments in one orchestra? Why the body of a double bass (=contrabas, the instrument on the very right in the photo with an arrow pointing to it) is so much larger than that of a violin (the dash arrow in the photo).

<u>Solution</u>:
This was a rather open question, which allowed for some freedom in your answers.
Of course, all students knew that the double bass is needed to produce very low notes, which you cannot get with a violin.
It is not only the question of how long your strings are. Also, the strings of a double bass are much thicker. Their relatively large mass (per unit length) make the waves slow. Smaller wave velocity *v* also make the frequency $\omega = \pi v / L$ lower.
The body of the instrument also plays a role. They resonate in a certain range of frequencies to enhance the sound and give a unique character to the sound of the notes.

<u>Problem 3.</u>

(**Question 3A**)
An interference pattern is created by two plane waves $A_1(\vec{r},t) = A_0 e^{i\omega t + i\vec{k}_1\cdot\vec{r}}$ and $A_2(\vec{r},t) = A_0 e^{i\omega t + i\vec{k}_2\cdot\vec{r}}$.
The wavevectors of these wave have Cartesian components $\vec{k}_1 = (k_x, k_y, 0)$ and $\vec{k}_2 = (k_x, -k_y, 0)$.
Calculate the intensity distribution $I(\vec{r}) = \left|A_1(\vec{r},t) + A_2(\vec{r},t)\right|^2$ in this interference pattern.

<u>Solution</u>:

$$I(\vec{r}) = \left|A_1(\vec{r},t) + A_2(\vec{r},t)\right|^2 = \left(A_0 e^{i\omega t + i\vec{k}_1\cdot\vec{r}} + A_0 e^{i\omega t + i\vec{k}_2\cdot\vec{r}}\right)\left(A_0^* e^{-i\omega t - i\vec{k}_1\cdot\vec{r}} + A_0^* e^{-i\omega t - i\vec{k}_2\cdot\vec{r}}\right)$$

$$= \left|A_0\right|^2 \left\{2 + e^{i\vec{k}_1\cdot\vec{r} - i\vec{k}_2\cdot\vec{r}} + e^{-i\vec{k}_1\cdot\vec{r} + i\vec{k}_2\cdot\vec{r}}\right\} = 2\left|A_0\right|^2 \left(1 + \cos(\vec{k}_1\cdot\vec{r} - \vec{k}_2\cdot\vec{r})\right)$$

By substituting $\vec{k}_1 = (k_x, k_y, 0)$ and $\vec{k}_2 = (k_x, -k_y, 0)$, one can find that the intensity is only a function of *y*:

$$I(\vec{r}) = I(y) = 2\left|A_0\right|^2 \left(1 + \cos(2k_y y)\right).$$

(**Question 3B**)
An interference pattern is created by two plane waves $A_1(\vec{r},t) = A_0 e^{i\omega_1 t + i\vec{k}_1\cdot\vec{r}}$ and $A_2(\vec{r},t) = A_0 e^{i\omega_2 t + i\vec{k}_2\cdot\vec{r}}$, where $\omega_1 \neq \omega_2$. Will these two waves interfere? Calculate the *time-averaged* intensity distribution $\bar{I}(\vec{r}) = \left\langle \left|A_1(\vec{r},t) + A_2(\vec{r},t)\right|^2 \right\rangle_t$, where the angular brackets $\langle ... \rangle_t$ denote averaging over time interval much larger than $2\pi / |\omega_1 - \omega_2|$.
<u>*Hint*</u>: The average value of cosine function is 0 ($\langle \cos\chi \rangle_\chi = 0$) if averaged over interval much larger than the period.

<u>Solution</u>:

$$I(\vec{r},t) = \left|A_1(\vec{r},t) + A_2(\vec{r},t)\right|^2 = \left(A_0 e^{i\omega_1 t + i\vec{k}_1\cdot\vec{r}} + A_0 e^{i\omega_2 t + i\vec{k}_2\cdot\vec{r}}\right)\left(A_0^* e^{-i\omega_1 t - i\vec{k}_1\cdot\vec{r}} + A_0^* e^{-i\omega_2 t - i\vec{k}_2\cdot\vec{r}}\right)$$

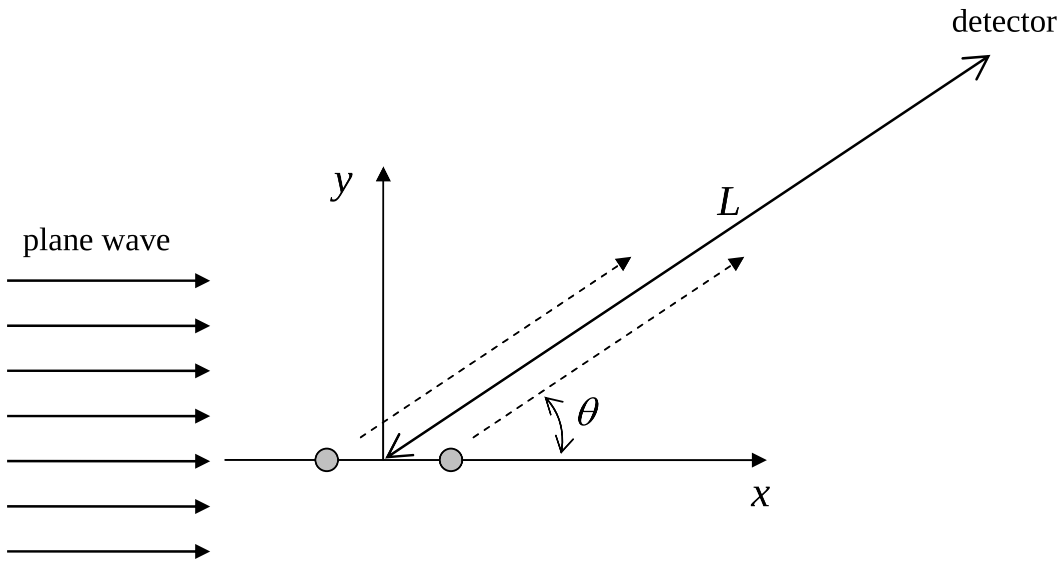
$$= \left|A_0\right|^2 \left\{2 + e^{i\omega_1 t - i\omega_2 t + i\vec{k}_1\cdot\vec{r} - i\vec{k}_2\cdot\vec{r}} + e^{-i\omega_1 t + i\omega_2 t - i\vec{k}_1\cdot\vec{r} + i\vec{k}_2\cdot\vec{r}}\right\} = 2\left|A_0\right|^2 \left(1 + \cos(\omega_1 t - \omega_2 t + \vec{k}_1\cdot\vec{r} - \vec{k}_2\cdot\vec{r})\right)$$

So, the waves will interfere (the second term in the result above). However, if $\omega_1 \neq \omega_2$, this interference term will disappear after time average of the intensity:

$$\bar{I}(\vec{r}) = \left\langle I(\vec{r},t)\right\rangle_t = 2\left|A_0\right|^2.$$

Problem 4.

Two identical point scatters are illuminated by a plane wave with wavelength λ travelling in the positive x direction. The scatters are placed at $\underline{r}_1 = (x_1,y_1,z_1) = (-a,0,0)$ and $\underline{r}_2 = (x_2,y_2,z_2) = (a,0,0)$. A detector is placed at distance L far from the scatters ($L >> 2a$) such that the direction to the detector makes an angle θ with the y axis.



(**Question 4A**)
Show that the amplitude of the wave at the detector can be written as

$$A = A_0 \frac{b}{L} e^{i\omega t - ik(L + a - a\cos\theta)} + A_0 \frac{b}{L} e^{i\omega t - ik(L - a + a\cos\theta)},$$

where A_0 is the amplitude of the incident wave and b is the scattering length.
Hint: Prove first that for $L >> 2a$ one can approximate the distances from the scatters to the detector as $L_1 \approx L + a\sin\theta$ and $L_2 \approx L - a\sin\theta$.

Solution:
I made a mistake in the formulation of this question. I made a typo in the Hint that I gave you. It should of course be:
"... as $L_1 \approx L + a\cos\theta$ and $L_2 \approx L - a\cos\theta$."
In checking your solutions, I have always interpreted your results in the best possible *for you* way.

At the detector position one has to sum the spherical waves scattered by both scatters:

$$A = A_0 \frac{b}{L_1} e^{-ika} e^{i\omega t - ikL_1} + A_0 \frac{b}{L_2} e^{-ik(-a)} e^{i\omega t - ikL_2},$$

where the $e^{-ik(-a)}$ and e^{-ika} phase factors appear since the incident wave arrives earlier to the scatter at $-a$ than to the one at a.

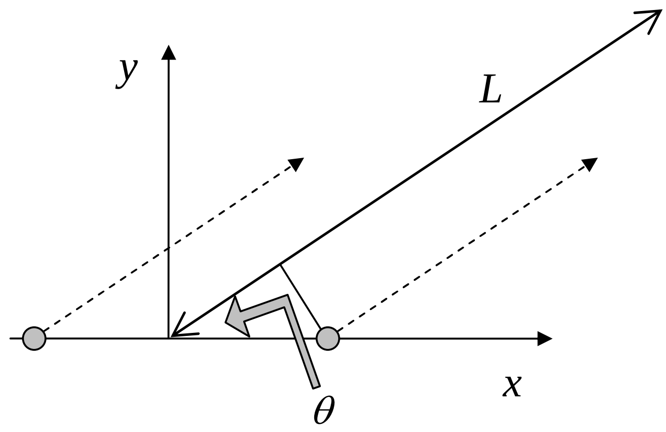
To calculate L_1 and L_2, one can use the cosine theorem:

$$(L_1)^2 = L^2 + a^2 + 2aL\cos\theta$$

$\left(L_2\right)^2 = L^2 + a^2 - 2aL\cos\theta$.

By neglecting the a^2 term and expanding the square root into the Taylor series, one get the desired relations: $L_1 \approx L + a\cos\theta$ and $L_2 \approx L - a\cos\theta$.

The same one can do by simply saying that the directions to the detector from the scatter 1 and 2 are practically parallel. The desired relations can be easily obtained using the figure:



Next step in the derivation is simply replacing ($1/L_1$) and ($1/L_2$) by ($1/L$) since this is a slowly-varying function. However, one has to take into account the terms linear in a in the fast-varying complex exponent. We arrive then at the desired relation:

$$A = A_0 \frac{b}{L} e^{i\omega t - ik(L + a - a\cos\theta)} + A_0 \frac{b}{L} e^{i\omega t - ik(L - a + a\cos\theta)}$$

(**Question 4B**)

Explain the $1/L$ dependence in the amplitudes of the waves in Question 4A.

Solution:

The $1/L$ dependence is related to the energy conservation law. The power of a spherical wave is spreading over a surface, which grows as L^2. This is compensated by the $1/L^2$ dependence in the intensity (power per unit area) $I \propto |A|^2$.

(**Question 4C**)

Calculate the intensity of the wave (see Question 4A) at the detector. At what value of θ one observes the first minimum of the intensity if

(i) a = 50 nm and λ = 0.1 nm (x-ray)

(ii) a = 50 nm and λ = 500 nm (visible light)

Solution:

$$I = \left|A\right|^2 = \left|A_0\right|^2 \frac{\left|b\right|^2}{L^2}\left\{1 + 1 + e^{i2k(a - a\cos\theta)} + e^{-i2k(a - a\cos\theta)}\right\} = 2\left|A_0\right|^2 \frac{\left|b\right|^2}{L^2}\left\{1 + \cos\left[2k(a - a\cos\theta)\right]\right\}$$

The minima are at $\cos\left[2k(a - a\cos\theta)\right] = -1$. The first minimum is therefore at $2k(a - a\cos\theta) = \pi$. Therefore, $a - a\cos\theta = \lambda/4$ or $\cos\theta = 1 - \lambda/(4a)$.

(i) By substituting the numbers, one gets $\cos\theta = 1 - 0.1/200 = 0.9995$ or $\theta = 1.8^{\circ}$

(ii) Here one finds that $\cos\theta = 1 - 500/200 = -1.5$. One cannot fulfill this condition and, therefore, the first minimum (when the waves scattered by the two objects cancel each other) cannot be reached.