Write down your name clearly on every sheet of paper!
You can give your answers either in Dutch or English.
Do not hesitate to ask if the question is not clear to you.

Golven tussentoets November 2011

Mathematics:

Geometrical series:

$$1+x+x^2+x^3+\ldots+x^{N-1}=\frac{1-x^N}{1-x}$$

$$1+x+x^2+x^3+\ldots=\frac{1}{1-x}$$

$$1-x+x^2-x^3+\ldots=\frac{1}{1+x}$$

Taylor expansions:

$$\sqrt{1\pm x}\simeq 1\pm\frac{1}{2}x \qquad (x\ll 1)$$

$$\sqrt{A\pm x}\simeq\sqrt{A}\sqrt{1\pm\frac{x}{A}}\simeq\sqrt{A}\left(1\pm\frac{x}{2A}\right)=\sqrt{A}\pm\frac{x}{2\sqrt{A}} \qquad (x\ll A)$$

Trigonometric relations:

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

Complex exponent:

$$e^{\pm i\phi}=\cos\phi\pm i\sin\phi$$

$$e^{i\phi}e^{-i\phi}=1$$

Problem 1.

(Question 1A)
Using the trigonometric relations given on the previous page prove that
$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$.

(Question 1B)
A wave is written using real functions as
$A_1(x,t) = A_0\sin(\omega t)\sin(kx)$

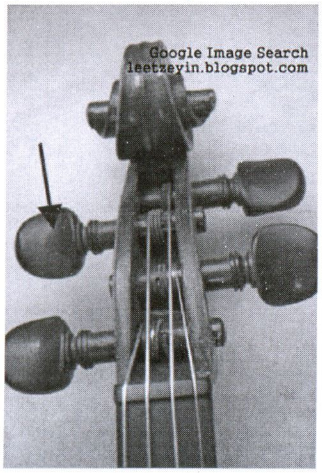
- Is it a standing or travelling wave? Motivate your answer!
- If it is a standing wave, what is its phase velocity?
  If it is a travelling wave, what is the distance between two neighbouring nodes?

(Question 1C)
Now, to the standing wave $A_1(x,t)$ from Question 1B, we add a travelling wave
$A_2(x,t) = \frac{1}{2}A_0\cos(\omega t + kx)$,

- Is $A_1(x,t) + A_2(x,t)$ a standing or travelling wave?
- If it is a standing wave, what is its phase velocity?
  If it is a travelling wave, what is the distance between two neighbouring nodes?

Problem 2.
(Question 2A)
Write down the formula for the lowest frequency ω (main tone) of a string with length L and the phase velocity of the string wave v.

(Question 2B)



A violin (=viool in Dutch) is tuned by turning the pegs (see the picture on the left).

Is it the length of the string L or the phase velocity v of the waves propagating along the string, which change during tuning?

On the other hand, while playing, the musician presses the string against the deck at different positions to produce different notes (see the photo on the right).

Explain why the tone depends on where the string is pressed.

Problem 3.

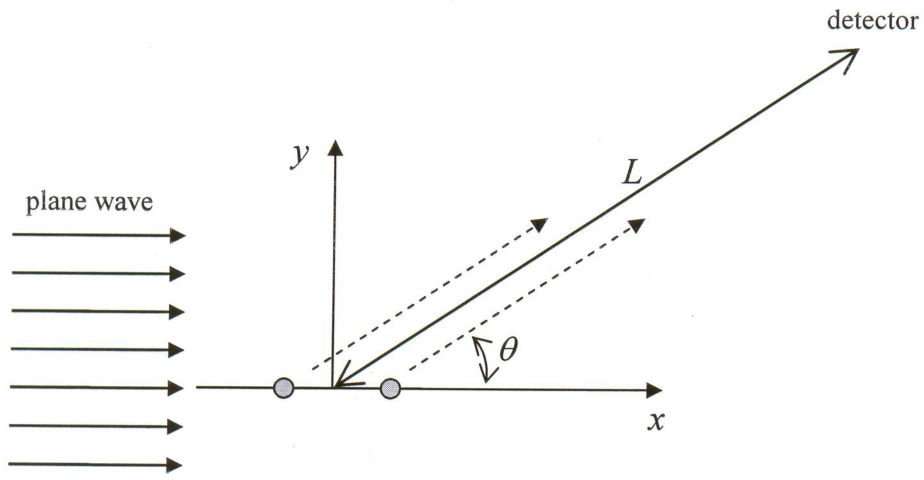
There are two plane waves $A_1(\vec{r},t) = A_0 e^{i\omega t + i\vec{k_1}\cdot\vec{r}}$ and $A_2(\vec{r},t) = A_0 e^{i\omega t + i\vec{k_2}\cdot\vec{r}}$. The wavevectors of these waves have Cartesian components $\vec{k_1} = (k_x, k_y, 0)$ and $\vec{k_2} = (k_x, -k_y, 0)$.

(Question 3A)
What is the phase velocity of the wave $A_1(\vec{r},t)$ and $A_2(\vec{r},t)$? Please give the absolute value and direction.

(Question 3B)
Calculate the (time-averaged) intensity distribution $I(\vec{r}) = \left\langle \left| A_1(\vec{r},t) + A_2(\vec{r},t) \right|^2 \right\rangle_t$ in the interference pattern created by the two waves.

Problem 4.

Two identical point scatters are illuminated by a plane wave with wavelength λ travelling in the positive x direction. The scatters are placed at $\underline{r}_1 = (x_1,y_1,z_1) = (-a,0,0)$ and $\underline{r}_2 = (x_2,y_2,z_2) = (a,0,0)$. A detector is placed at distance L far from the scatters ($L >> 2a$) such that the direction to the detector makes an angle θ with the y axis.



(Question 4A)
Show that the amplitude of the wave at the detector can be written as

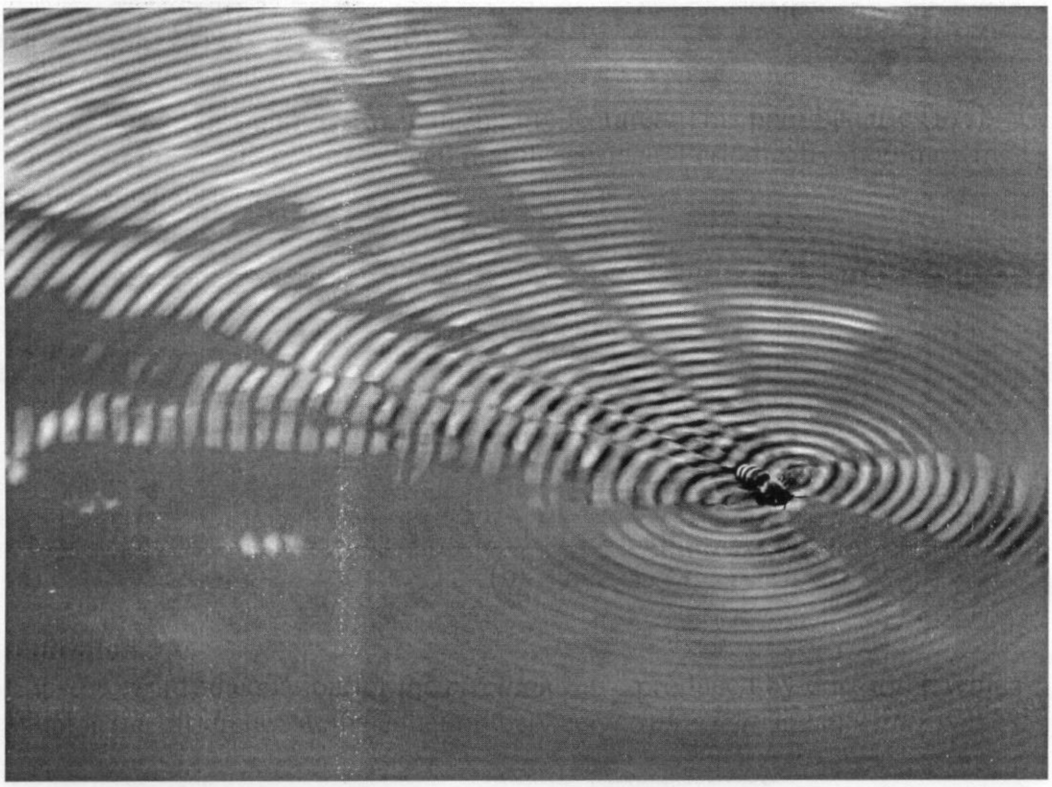
$$A = A_0 \frac{b}{L} e^{i\omega t - ik(L + a - a\cos\theta)} + A_0 \frac{b}{L} e^{i\omega t - ik(L - a + a\cos\theta)},$$

where A_0 is the amplitude of the incident wave and b is the scattering length.
Hint: Prove first that for $L >> 2a$ one can approximate the distances from the scatters to the detector as $L_1 \approx L + a\cos\theta$ and $L_2 \approx L - a\cos\theta$.

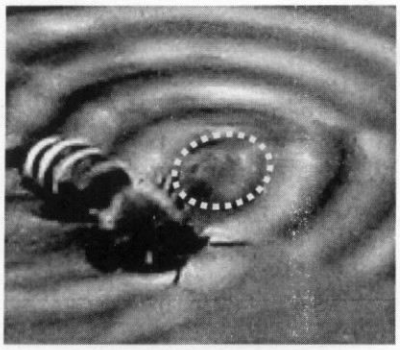
(Question 4B)
Explain the $1/L$ dependence in the amplitudes of the waves in Question 4A.

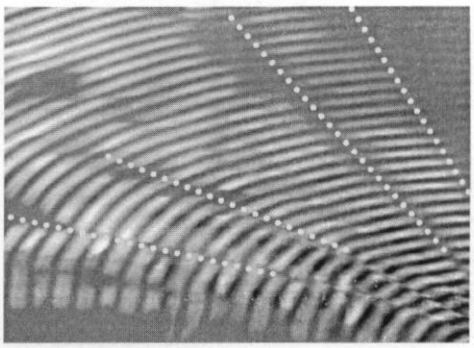
(Question 4C)
Calculate the intensity of the wave in Question 4A at the detector.

Problem 5.



I have showed this photo to you during my lectures. This poor bee tries to fly away using its wings but the capillary forces keep it at the surface. Periodically moving wings produce waves on the water surface.

(a) 

(b) 

(Question 5A)
Can you write an equation for the ***circular*** wave produced by one of the wings? [see Fig. (a)]. What is the difference in the equation between a ***spherical*** and ***circular*** wave?

(Question 5B)
Explain why the wave 'disappears' in certain directions [see the dotted lines in Figure (b) above].