Problem 1

A string of a guitar is tuned to produce an "A" note with a period of T_1 = (1/440) s.

(**Question 1A**)
What is the wave velocity in the string? Assume that the string is 1 meter long.

Answer:

The string resonance occurs when the length of a string L is equal to half-integer times the wavelength λ, i.e. $L = n\lambda/2$. The main tone corresponds to $n = 1$. Thus, $L = \lambda/2$. The phase velocity of the wave in the string is therefore $v = \omega/k = \lambda/T = 880$ m/s.

(**Question 1B**)
What is the wavelength of the spherical sound waves produced by the guitar in air (sound velocity in air is 330 m/s)?

Answer:

This is simple: $\lambda = vT = 0.75$ m.

Problem 2: use trigonometric functions!

(**Question 2A**)
Using the trigonometric formulas on the first page, prove that

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

Answer:

One can, for example, use the following relation:

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

By substituting $x = \frac{\alpha+\beta}{2}$; $y = \frac{\alpha-\beta}{2}$ and, therefore, $\alpha = x + y$; $\beta = x - y$, one gets the requested relation.

Two waves are written as

$$A_1(x,t) = \frac{1}{2}A_0\cos(\omega t + kx)$$

$$A_2(x,t) = A_0 \sin kx \sin \omega t$$

(**Question 2B**)

- Which of the waves, $A_1(x,t)$ or $A_2(x,t)$, is a standing wave? Motivate your answer! Calculate the positions of the nodes in this standing wave.

Answer:

$A_2(x,t)$ is a standing wave. It possesses nodes where $\sin kx = 0$ and the amplitude is therefore zero at any time t. The nodes are therefore observed at $kx = \pi n$, where *n* is an integer. This defines the node positions: $x = \lambda n/2$.

(**Question 2C**)

We now add the two waves together:

$$A_3(x,t) = A_1(x,t) + A_2(x,t)$$

What type of wave (travelling/standing/…) do we get as a result of their interference? Motivate your answer!

Answer:

Using the formula given in Question 1A, one can rewrite

$$A_2(x,t) = A_0 \sin kx \sin \omega t = A_0/2 \left[\cos(kx - \omega t) - \cos(kx + \omega t)\right]$$

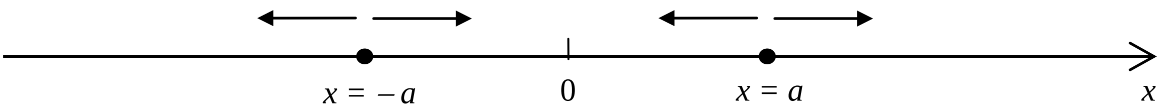
By adding $A_1(x,t)$, one gets

$$A_1(x,t) + A_2(x,t) = \frac{A_0}{2}\left[\cos(kx + \omega t) + \cos(kx - \omega t) - \cos(kx + \omega t)\right] = \frac{A_0}{2}\cos(kx - \omega t)$$

.

This is a travelling wave but moving in the positive *x*-direction now: as time increases one has to increase *x* to stay at the same phase (e.g., at the maximum amplitude).

Problem 3: use complex notations!



On the *x* axis there are two identical sources of one-dimensional waves located at $x = -a$ and $x = a$.

(**Question 3A**)

Argue that the total amplitude of the wave at $x > a$ can be written as

$$A(x,t) = 2A_0\, e^{i\omega t - ikx} \cos ka$$

Answer:

In this part of the *x*-axis one has two waves present, both travelling to the right. However, the distance to be travelled from the source to the observation point (i.e., *x*) is different for the two waves: *x* + *a* for the left source and *x* - *a* for the right source. The amplitude can be written as

$$A(x,t) = A_0\, e^{i\omega t - ik(x-a)} + A_0\, e^{i\omega t - ik(x+a)} = A_0\, e^{i\omega t - ikx}\left(e^{ika} + e^{-ika}\right) = 2A_0\, e^{i\omega t - ikx} \cos ka\ .$$

This equation describes a wave travelling to the right (see the $e^{i\omega t - ikx}$ part) with the amplitude that depends on the distance 2*a* between the two sources.

(**Question 3B**)

Argue that the total amplitude of the wave at $-a < x < a$ can be written as

$$A(x,t) = 2A_0\, e^{i\omega t - ika} \cos kx$$

Answer:

In this part of the *x*-axis one has two waves present, which travel in opposite directions. Moreover, the distance to be travelled from the source to the observation point (i.e., *x*) is different for the two waves: *x* + *a* for the left source and *x* - *a* for the right source. The amplitude can be written as

$$A(x,t) = A_0\, e^{i\omega t + ik(x-a)} + A_0\, e^{i\omega t - ik(x+a)} = A_0\, e^{i\omega t - ika}\left(e^{ikx} + e^{-ikx}\right) = 2A_0\, e^{i\omega t - ika} \cos kx\,.$$

This equation describes a standing wave (*x*-dependence is decoupled from time dependence). The e^{-ika} part is a complex number describing the starting phase of this standing wave, at *t* = 0).

(**Question 3C**)

Calculate the intensity of the wave $I(x)$ for $x > a$.

Answer:

One can, e.g., use the result of 3A:

$$I(x) = 2A_0\, e^{i\omega t - ikx} \cos ka \times 2A_0^*\, e^{-i\omega t + ikx} \cos ka = 4\left|A_0\right|^2 \cos^2 ka$$

(**Question 3D**)

Calculate the intensity of the wave $I(x)$ for $-a < x < a$.

Answer:

Using the result of 3B:

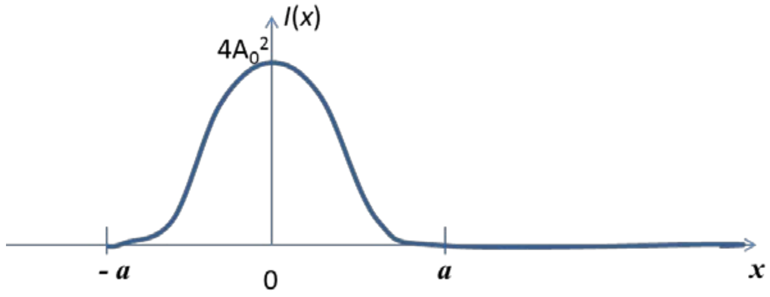
$$I(x) = 2A_0\, e^{i\omega t - ika} \cos kx \times 2A_0^*\, e^{-i\omega t + ika} \cos kx = 4\left|A_0\right|^2 \cos^2 kx$$

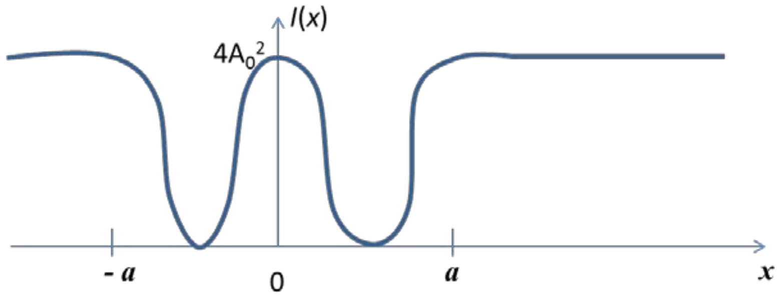
(**Question 3E**)

Schematically draw the $I(x)$ graph for

- $a = \lambda / 4$,
- $a = \lambda / 2$.

Answer:




I(x)
4A0 2
- a
0
a
x

X-ray, neutron and light scattering

1. What is the meaning of the radius of gyration R_g? How can it be determined from scattering data? What range of scattering wavevectors should be used to determine R_g for particles with diameter of about 10 nm (give an order-of-magnitude estimate)? [2 points]

[both K here and q in problem 2 denote the same quantity, the scattering wavevector] The radius of gyration R_g is a measure of the typical size of a scattering object. It equals to the mean-square distance from the centre of the objected weighted with the scattering length.

The radius of gyration R_g can be determined by fitting a straight line to the dependence of the logarithm of the scattering intensity versus K^2 for the smallest K-values. The fit must be performed in the Guinier range, $KR_g \ll 2\pi$, at low concentration of the scattering objects so that there is negligible correlation between their positions. For particles of the order of 10 nm the values of the scattering wavector must therefore be $K \ll 2\pi / 10$ nm, or $K \sim 0.1 \text{nm}^{-1}$.

Not explicitly requested but possible to add:

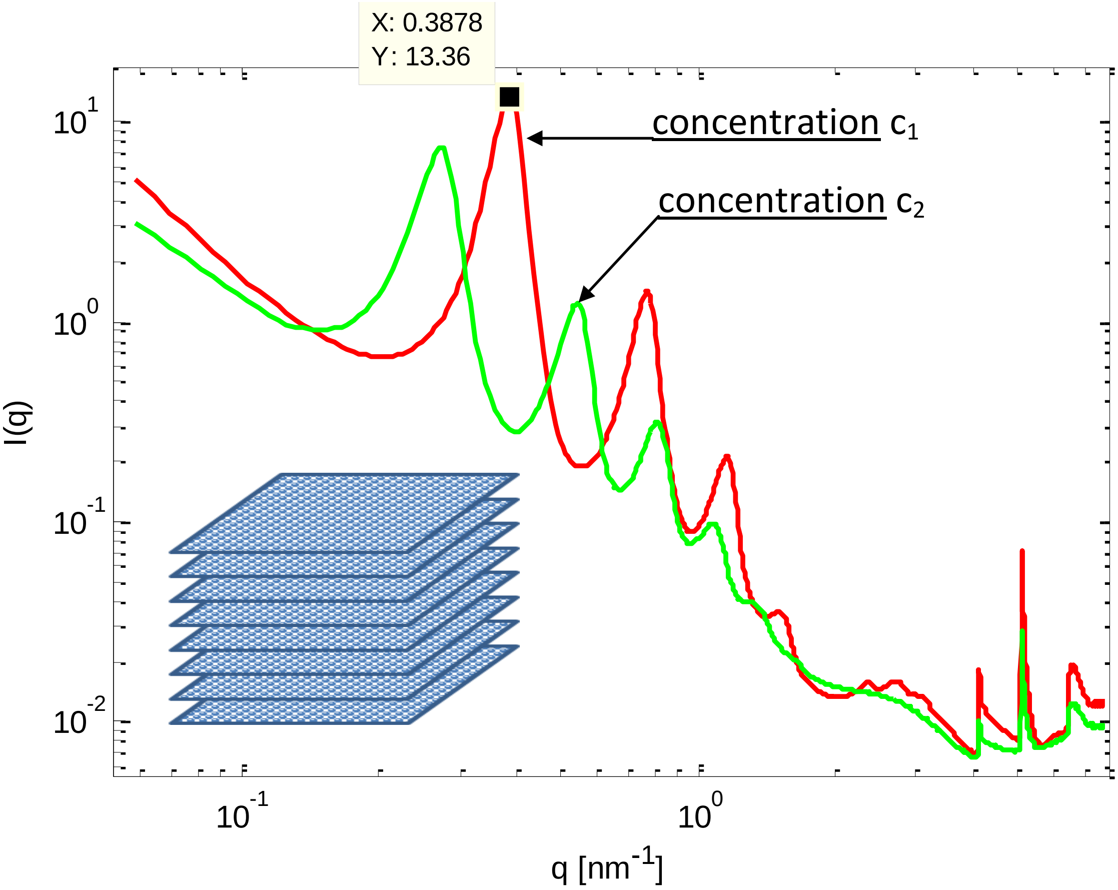
The mathematical definition of the radius of gyration $R_g^2 = \sum_i b_i (\vec{r} - \vec{R}_b)^2 / \sum_i b_i$, where $\vec{R}_b = \sum_i b_i \vec{r} / \sum_i b_i$.

Guinier law $I(K) = I(K \to 0)\exp[-\frac{1}{3} K^2 R_g^2]$.

2. The figure below displays the small-angle x-ray scattering intensity $I(q)$ as a function of the scattering wavevector q for microtubles self-assembled in solution of sodium dodecyl sulfate (SDS) and β-cyclodextrin (β-CD) in water with two different concentrations. The saw-tooth shaped peaks at scattering wavevectors q > 4 nm^{-1} are related with the 2-dimentional (2D) structure of bilayers of SDS@2β-CD complexes while the peaks at q < 4 nm^{-1} originate from periodic arrangement of the 2D bilayers into multilayer stacks as schematically illustrated in the inset. The x-ray wavelength was λ = 0.1 nm.

   a) These data are taken between q_{min} = 0.06 nm^{-1} and q_{max} = 7.5 nm^{-1}. What range of scattering angles θ was used in this experiment?

Using the formula $q = \frac{4\pi}{\lambda}\sin\frac{\theta}{2}$, one can easily calculate that $\theta_{min} = 1$ mrad = 0.055^o and $\theta_{max} = 0.12$ rad = 6.8^o.



b) Does the distance between SDS@2β-CD complexes within the 2D bilayers depend on the concentration? [see data at $q > 4$ nm^{-1}]

No, the position of the peaks at $q > 4$ nm^{-1} is the same for both concentrations and, therefore, the distance must remain the same.

c) Does the distance between the bilayers change with concentration? [data at $q < 4$ nm^{-1}]

Yes, the peaks at $q < 4$ nm^{-1}obviously shift with concentration indicating a variation of the interlayer distance.

d) For sample with concentration c_1 the first peak is observed at $q = 0.39$ nm^{-1}. What is the distance between the bilayers in the stack?

The distance between bilayers in a lamella structure is simply
$d = 2\pi / q = 16\,\text{nm}$.

e) Is this distance larger or smaller at concentration c_2? Can you guess which of the two concentrations is higher?

At concentration c_2 the peaks appear at <u>lower</u> q values indicating a <u>larger</u> interlayer distance. This can be related with swelling of the lamella structure. Thus, c_2 is presumably lower than c_1.

f) Can light scattering be used to reveal these structures? Calculate the maximum value of the scattering wavevector q, which can be reached using visible light. Assume that the wavelength of light is λ = 633 nm (HeNe laser) and the average refractive index of the sample is n = 1.4.

No, these structures have a too small period to be probed by static light scattering. For light scattering one needs to take the average refractive index n into account to calculate the scattering wavevector: $q = \frac{4\pi}{\lambda} n \sin\frac{\theta}{2}$. The maximum q value can be reached for $\theta = \pi = 180^\circ$. Thus, $q_{\max} = \frac{4\pi}{\lambda} n = 0.028$ nm^{-1}. This value is two times smaller than the minimum q-value shown in the graph.

g) Argue that small-angle neutron scattering using neutrons with λ = 0.3 nm can also be used to reveal these structures. Why would one like to do that? What are advantages and disadvantages of SANS in comparison with SAXS for this system?

The same range of q-values can be reached using neutrons with λ = 0.3 nm although the maximum scattering angle has to be increased up to $\theta_{\max} = 20.6°$.
[<u>Advantage:</u>] SANS data could complement SAXS data because the scattering length density profile for neutrons is different than that for SAXS. More importantly, this profile can be varied using isotope H/D substitution. This variation can reveal further details of the structure.
[<u>Disadvantages:</u>] However, the quality of SANS data could be deteriorated because of inelastic scattering of neutrons (=high incoherent background). Moreover, much lower intensity of the neutron sources significantly increase the measurement time limiting studies of the response of the structure to various parameters [*e.g.*, time, concentration, temperature, ionic strength].

Correct and <u>well-argumented</u> answers to questions a) – e) can give you up to 1 point each; questions f) and g) are up to 1.5 points each. The total maximum number of points for questions 1 and 2a—2g is therefore 2 + 5*1 + 2*1.5 = 10.

Optical microscopy

a) 60x/1.4

$$d_{ax} = 1.4\frac{\lambda}{n\sin^2\alpha} = 1.4\frac{n\lambda}{NA^2} = 1.4\frac{1.55•0.532}{1.4^2} = 0.59\mu m$$

$$d_{lat} = 0.45\frac{\lambda}{NA} = 0.45\frac{0.532}{1.4} = 0.17\mu m$$

Note : this cannot be an air or water immersion objective since NA = n.sin(alpha) = 1.4 → n = NA/sin(alpha) and sin(alpha) max = 1. So this is an oil immersion objective.

b) $Absorbed\_fraction = 1 - I/I_0 = \mu C d_{ax} = 85000\cdot10^{-5}\cdot0.59\cdot10^{-4} = 5.0\cdot10^{-5}$

Absorbed energy/s = $25\cdot10^{-6}\cdot5\cdot10^{-5} = 1.25\cdot10^{-9}J$

1 photon = $3eV\cdot1.6\cdot10^{-19}J/eV = 4.8\cdot10^{-19}J/photon \rightarrow$

$$1.25\cdot10^{-9}J \Big/ 4.8\cdot10^{-19}J/photon = 2.6\cdot10^{9}\,photons$$

Volume ~ $d_{lat}^2•d_{ax} = 0.17^2•0.59 = 0.017\mu m^3 = 1.7•10^{-17}l$

10uM . volume. N_a → ~102 molecules

Number of excitations/molecule/s = $2.6\cdot10^{9}\big/102 = 2.5\cdot10^{7}$

c) $\sigma = 1000\cdot\mu/N_a = 1.4\cdot10^{-16}cm^2$

d) see lecture slides

e)) $I_{peak} = \frac{T}{\Delta\tau}\cdot\frac{P}{\pi(d_{airy}/2)^2} = \frac{12.2\cdot10^{-9}}{50\cdot10^{-15}}\cdot\frac{5\cdot10^{-3}}{\pi\cdot(0.45\cdot10^{-4})^2} = 1.9\cdot10^{11}W/cm^2$

f) spatial extend = $50\cdot10^{-15}\cdot3\cdot10^{8}m/s = 15\mu m$

15um/1.024um ~ 15 oscillations

ASSM 2015-2016 – Question EM – Answers

a. (4 points). Consider a photon with an energy of 100 eV, and an electron with an energy of 100 eV, both travelling in vacuum. For each particle, give the wavelength and the velocity. (N.B. Your answer should contain 4 numbers).

Answer: Photon: $\lambda=hc/E=1.24\cdot10^{-8}$ m, $c=3.00\cdot10^{8}$ m/s. Electron: $v=\sqrt{(2E/m)}=5.93\cdot10^{6}$ m/s, $\lambda=h/p=1.23\cdot10^{-10}$ m.

b. (5 points). Give 3 types of contrast that can generally occur when imaging in bright-field TEM imaging mode. (no explanation required)

Answer: diffraction contrast, phase contrast, mass-thickness contrast

c. (6 points) Mention the imaging mode that generates so-called Z-contrast, and explain what kind of detector has to be used for this purpose.

Answer: High angle annular dark field scanning transmission electron microscopy (HAADF-STEM) generates Z-contrast with relationship for the intensity $I\sim Z^2$, where Z is the atomic number. This is achieved by using an annular-ring shaped detector below the sample, whereby electrons that have scattered over large angles (typically 4-10°) are detected.

d. (4 points) Explain briefly how a magnet can be used to separate electrons having different energies when performing EELS.

*After interaction with the samples, a number of electrons in the electron beam have lost energy. By placing a magnet at the bottom of the column with the magnetic field perpendicular to that column, the electrons are deflected by the Lorentz Force, $\mathbf{F}=q\,\mathbf{v}\times\mathbf{B}$. Here **F, v,** and **B** are vectors. The velocity **v** depends on the energy of the electrons. Because the velocity depends on the electron energy through $E=\frac{1}{2}mv^2$, electrons with different energies are deflected differently and can thus be resolved in a spectrum.*

e. (6 points) Consider the figures below, showing the results of an *in-situ* experiment whereby MgO nanoparticles react with H_2O and are transformed into $Mg(OH)_2$ upon exposure to water vapor. The circled areas in panels (a,b) show where the electron beam was during the experiment, which lasted 2400s and during which the electron beam was continuously on. From an analysis of the TEM images and of the diffraction patterns that were recorded during the experiment, discuss what happens to the crystal structure of the nanoparticles during the chemical transition, and whether the electron beam has had an influence on this process.

Answer: From the diffraction patterns, it is clear that at first, well-defined diffraction spots are observed whereas at the end of the experiment, only diffuse rings are present. Therefore, the nanoparticles transform from atomically crystalline MgO to amorphous $Mg(OH)_2$. (In addition, diffraction contrast is visible in TEM image (A) which is absent in TEM image (B)). From the TEM images (A,B) it is also clear that areas that were outside the electron beam during the long-time exposure are much less transformed. Therefore, the electron beam has had a strong and determining influence on the transition.