Answers for the Waves test, November 2016

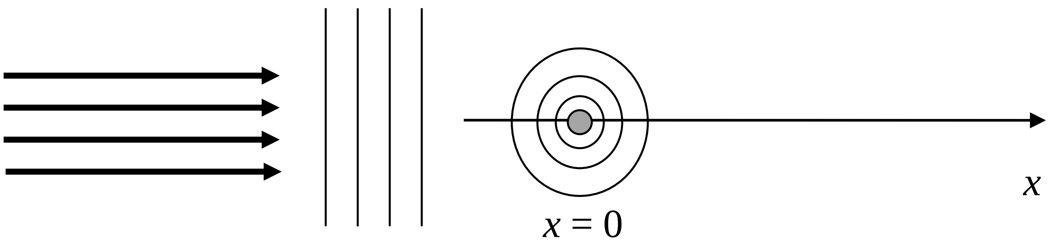
(**Question 1A**)
These waves possess no nodes, i.e., *x* values at which the amplitude *A* is zero for all *t*. They are, instead, travelling waves.
For the first wave $A_1(x,t) = 2A_0\cos(-\omega t + kx)$: At any subsequent time instance $t+\Delta t$ the wave profile remains the same but is shifted in space:
$A_0\cos\left[-\omega(t+\Delta t)+k(x+\Delta x)\right] = A_0\cos(-\omega t + kx)$ for all *x* and *t*. This condition is fulfilled for $k\Delta x - \omega\Delta t = 0$. Therefore, the phase velocity is $v = \Delta x/\Delta t = \omega/k$. This wave travels in the *positive x*-direction.
Similarly, for the second wave $A_2(x,t) = 2A_0\cos(-\omega t - kx)$:
$A_0\cos\left[-\omega(t+\Delta t)-k(x+\Delta x)\right] = A_0\cos(-\omega t - kx)$ must hold for all *x* and *t*. This condition is fulfilled for $-k\Delta x - \omega\Delta t = 0$. Therefore, the phase velocity is $v = \Delta x/\Delta t = -\omega/k$. This wave travels in the *negative x*-direction.

(**Question 1B**)
Using trigonometric relations provided, one can easily derive

$$A_1(x,t) + A_2(x,t) = A_0\left[\cos(-\omega t + kx) + \cos(-\omega t - kx)\right]$$
$$= 2A_0\cos(-\omega t)\cos(kx) = 2A_0\cos(\omega t)\cos(kx)$$

This is a standing wave since there are *x* values where the amplitude is equal to 0 for all *t*. The position of these nodes is governed by the relation $\cos(kx) = 0$, which is satisfied at $kx = \pi/2 + n\pi$. Thus, the positions of the nodes is $x = (\pi/2 + n\pi)/k = \lambda/4 + n\lambda/2$, where $n = 0, \pm1, \pm2, \dots$

(**Question 2A**)



The incident plane wave is written as $A_0 e^{i\omega t - ikx}$ (the first term). The scattered wave can be written as $A_0 \frac{b}{r} e^{i\omega t - ikr}$. For the points on the *x* axis $r = |x|$, i.e., $r = x$ for *x* > 0 and $r = -x$ for *x* < 0. Thus, the sum of the incident wave and the wave scattered by the second atom can be written as

$A(x,t) = A_0 e^{i\omega t - ikx} + A_0 \frac{b}{x} e^{i\omega t - ikx}$ for *x* > 0 and

$A(x,t) = A_0 e^{i\omega t - ikx} + A_0 \frac{b}{x} e^{i\omega t + ikx}$ for *x* < 0.

(**Question 2B**)

After a simple calculation, one gets (assuming *b* is real)

$$I(x=0)=\left(A_0 e^{i\omega t-ikx}+A_0\frac{b}{x}e^{i\omega t+ikx}\right)\left(A_0^* e^{-i\omega t+ikx}+A_0^*\frac{b}{x}e^{-i\omega t-ikx}\right)$$

$$=|A_0|^2\left[1+\frac{b^2}{x^2}+\frac{b}{x}\left(e^{2ikx}+e^{-2ikx}\right)\right]=|A_0|^2\left(1+2\frac{b}{x}\cos(kx)+\frac{b^2}{x^2}\right)$$

(**Question 3A**)

At the detector position one has to sum the spherical waves scattered by both scatters:

$$A=A_0\frac{b}{L_1}e^{i\omega t-ikL_1}+A_0\frac{b}{L_2}e^{i\omega t-ikL_2}.$$

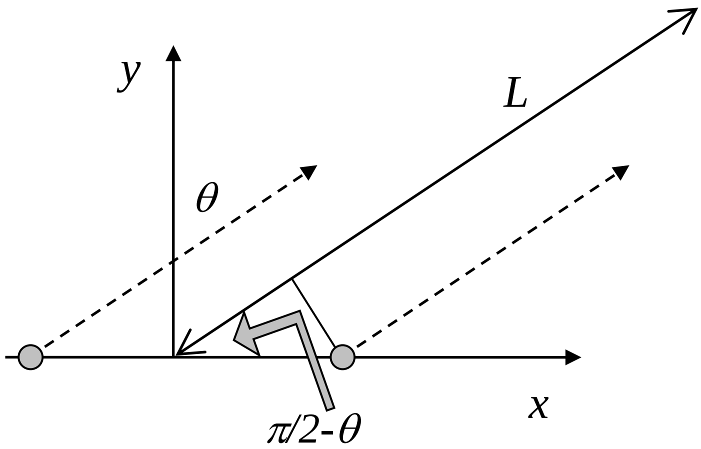
To calculate L_1 and L_2, one can use the cosine theorem:

$$(L_1)^2=L^2+a^2-2aL\cos\left(\frac{\pi}{2}+\theta\right)=L^2+a^2+2aL\sin\theta$$

$$(L_2)^2=L^2+a^2-2aL\cos\left(\frac{\pi}{2}-\theta\right)=L^2+a^2-2aL\sin\theta.$$

By neglecting the a^2 term and expanding the square root into the Taylor series, one get the desired relations: $L_1 \approx L + a\sin\theta$ and $L_2 \approx L - a\sin\theta$.

The same one can do by simply saying that the directions to the detector from the scatter 1 and 2 are practically parallel. The desired relations can be easily obtained using the figure:



Next step in the derivation is simply replacing $(1/L_1)$ and $(1/L_2)$ by $(1/L)$ since this is a slowly-varying function. However, one has to take into account the terms linear in *a* in the fast-varying complex exponent. We arrive then at the desired relation:

$$A=A_0\frac{b}{L}e^{i\omega t-ik(L-a\sin\theta)}+A_0\frac{b}{L}e^{i\omega t-ik(L+a\sin\theta)}$$

(**Question 3B**)

$$I=|A|^2=|A_0|^2\frac{|b|^2}{L^2}\left\{1+1+e^{i2ka\sin\theta}+e^{-i2ka\sin\theta}\right\}=2|A_0|^2\frac{|b|^2}{L^2}\left\{1+\cos\left[2ka\sin\theta\right]\right\}$$

One get a minimum when $\cos[2ka\sin\theta] = -1$. The first minimum is at $2ka\sin\theta = \pi$ or $\sin\theta = (1/4)(2\pi/k)(1/a) = \lambda/4a$. With the numbers given $\sin\theta = 0.0005$ or $\theta = 0.0005 \text{ rad} = 0.029^\circ$.

(**Question 4A**)

One needs to explain two points here:

(a) The amplitude of the waves reduces with the distance since the wave is spreading over an increasing area. The total power *P* of the wave can be calculated as its intensity at a distance *r*, proportional to $|A(\vec{r},t)|^2$, times the perimeter $2\pi r$ of the circle. To ensure the energy conservation law the total power *P* should be independent of the choice of *r*. Thus, the amplitude must reduce as $1/\sqrt{r}$.

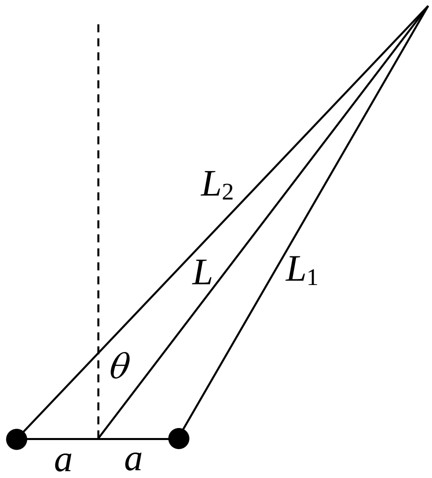
(b) A circular wave propagates in all direction along the surface. Therefore, its phase must be the function of the distance travelled but not of the direction. Moreover, the phase $\omega t - kr$ propagates outwards: with increasing t one has to increase r to stay at the same phase in the wave. The complex exponent $e^{i\omega t - ikr}$ perfectly satisfies this condition.

(**Question 4B**)

Similar to **3A**,

$$(L_1)^2 = L^2 + a^2 - 2aL\cos\left(\pi/2 - \theta\right) = L^2 + a^2 - 2aL\sin\theta$$

$$(L_2)^2 = L^2 + a^2 - 2aL\cos\left(\pi/2 + \theta\right) = L^2 + a^2 + 2aL\sin\theta$$



By neglecting the a^2 terms and using the Taylor expansion $\sqrt{1+x} \approx 1 + x/2$ for small *x*, one gets

$$L_1 \approx L - a\sin\theta$$

$$L_2 \approx L + a\sin\theta$$

which can be used in the exponential function.

Moreover, in the denominators one can simply replace L_1 and L_2 by *L*.

Now, one can sum the amplitude and rewrite the result such that the effect of the interference is explicit.

Alternatively, one can directly calculate the intensity as follows.

$$I = A(\vec{r},t)A^*(\vec{r},t)$$

$$= \left(\frac{A_0}{\sqrt{L}}e^{i\omega t - ikL + ika\sin\theta} + \frac{A_0}{\sqrt{L}}e^{i\omega t - ikL - ika\sin\theta}\right)\left(\frac{A_0^*}{\sqrt{L}}e^{-i\omega t + ikL - ika\sin\theta} + \frac{A_0^* s}{\sqrt{L}}e^{-i\omega t + ikL + ika\sin\theta}\right)$$

$$= \frac{|A_0|^2}{L}\left(1 + 1 + e^{2ika\sin\theta} + e^{-2ika\sin\theta}\right) = 2\frac{|A_0|^2}{L}\left[1 + \cos(2ka\sin\theta)\right]$$

The destructive interference will lead to zero intensity when $\cos(2ka\sin\theta) = -1$. This happens when $2ka\sin\theta = \pi + 2\pi n$ or $\sin\theta = (\pi + 2\pi n)/(2ka) = 0.25 + 0.5n$ (according to the numbers given, $2ka = 4\pi$). Since $|\sin\theta| \le 1$, possible values of $\sin\theta$ are ± 0.25 and ± 0.75. Thus, destructive interference will be observed for $\theta \approx \pm 14.5^\circ, \pm 48.6^\circ$.