<u>Problem 1:</u> using trigonometric functions

(**Question 1A**)

Using the trigonometric formulas on the first page, prove that

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

One can, for example, use the following relation:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

By substituting $x = \frac{\alpha+\beta}{2}$; $y = \frac{\alpha-\beta}{2}$ and, therefore, $\alpha = x+y$; $\beta = x-y$, one gets the requested relation.

Two waves are written as

$$A_1(x,t) = -A_0\cos(\omega t + kx)$$
$$A_2(x,t) = A_0\cos kx \cos \omega t$$

(**Question 1B**)

Which of the waves, $A_1(x,t)$ or $A_2(x,t)$, is a travelling wave? What is its phase velocity and in which direction the wave propagates?

Which of the waves, $A_1(x,t)$ or $A_2(x,t)$, is a standing wave? Calculate the positions of the nodes in this standing wave.

$A_1(x,t)$ is a travelling wave since as time progresses from t to $t+\Delta t$, the phase of the wave will remain the same if one moves from x to $x-\omega\Delta t/k$. This wave is therefore propagates in the *negative* x-direction with the speed of $-\omega/k$.

$A_2(x,t)$ is a standing wave. It possesses nodes where $\cos kx = 0$ and the amplitude is therefore zero at any time t. The nodes are therefore observed at $kx = \pi/2 + \pi n$, where *n* is an integer. This defines the node positions: $x = \lambda/4 + \lambda n/2$.

(**Question 1C**)

We now add the two waves together:

$$A_3(x,t) = A_1(x,t) + A_2(x,t)$$

What type of wave (travelling/standing/…) do we get as a result of their interference?

<u>Hint</u>: write the standing wave as a sum of two counter-propagating travelling waves.

Using the formula given in Question 1A, one can rewrite

$$A_2(x,t) = A_0\cos kx\cos\omega t = A_0/2\left[\cos(kx+\omega t)+\cos(kx-\omega t)\right]$$

By adding $A_1(x,t)$, one gets

$$A_1(x,t) + A_2(x,t) = A_0/2\left[-\cos(kx+\omega t)+\cos(kx-\omega t)\right] = A_0\sin kx \sin\omega t.$$

This is a standing wave again but with different positions of the nodes.

<u>Problem 2.</u> using complex functions

Two waves with a period T and a wavelength λ are written as

$$A_1(x,t) = A_0 e^{i\omega t - ikx}$$

$$A_2(x,t) = A_0 e^{i\omega t + ikx + i\chi}$$

(Question 2A):

What type of the waves are they? Motivate your answer!

What is the relation between ω, k, T and λ?

Both waves are travelling waves since as time progresses from t to $t + \Delta t$, one can restore the same amplitude by shifting x in the *positive* (for A_1) or *negative* (for A_2) direction along the x-axis.

The requested relations are: $\omega = 2\pi / T$ and $k = 2\pi / \lambda$.

(Question 2B):

Prove that the intensity of the interference pattern between the two waves $A_1(\vec{r},t) + A_2(\vec{r},t)$ can be written as

$$I(x) = 2|A_0|^2 (1 + \cos(2kx + \chi))$$

$$I(x) = |A_1(x,t) + A_2(x,t)|^2 = \left(A_1(x,t) + A_2(x,t)\right)\left(A_1^*(x,t) + A_2^*(x,t)\right) =$$

$$\left(A_0 e^{i\omega t - ikx} + A_0 e^{i\omega t + ikx + i\chi}\right)\left(A_0^* e^{-i\omega t + ikx} + A_0^* e^{-i\omega t - ikx - i\chi}\right) = |A_0|^2 \left(1 + 1 + e^{2ikx + i\chi} + e^{-2ikx - i\chi}\right) =$$

$$2|A_0|^2 (1 + \cos(2kx + \chi))$$

(Question 2C):

at what value of χ there will be a node at $x = 0$?

Node means $A = 0$ at all times or simply $I = 0$. Using the result of (2B) one can see that $I = 0$ when $\cos(2kx + \chi) = -1$. A node will be observed at $x = 0$ when $\chi = \pi$.

Some of you have also given $\chi = \pi + 2\pi n$, *n = any integer, which is also fine.*

(Question 2D):

What is the distance between the nodes?

If one of the nodes is observed at a certain x value, the next one will take place when $2kx$ changes by 2π. The distance between the nodes is therefore

$\Delta x = (2\pi) / (2k) = \pi / k = \lambda / 2$.

Problem 3.

Two identical atoms are located at $x = 0$ and $x = a$. They are irradiated by a plane wave travelling along the *x*-axis in the positive *x*-direction. Each atom can weakly scatter the wave and their scattering length is *b*.

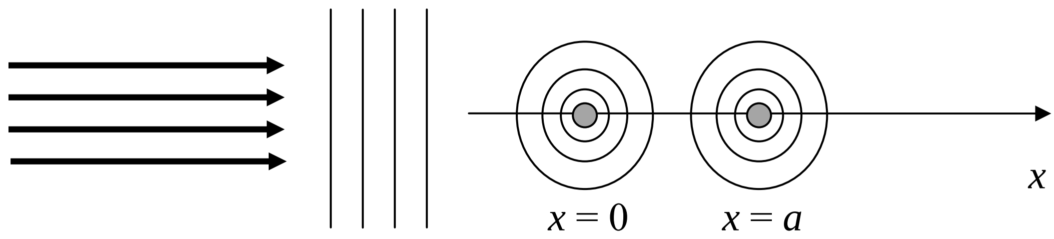
(**Question 3A**)

Make a sketch of the situation.

Argue that the total amplitude of the wave at the position of atoms 1 and 2 can be written as

$$A_1(x=0,t) = A_0 e^{i\omega t} + A_0 \frac{b}{a} e^{i\omega t - 2ika}$$

$$A_2(x=a,t) = A_0 e^{i\omega t - ika} + A_0 \frac{b}{a} e^{i\omega t - ika}$$



Each atoms experiences the incident wave and the wave scattered by another atom (NOT its own scattered wave!)

Incident plane waves (first terms) have different phases at positions $x = 0$ and $x = a$: $A_0 e^{i\omega t}$ and $A_0 e^{i\omega t - ika}$, respectively.

Scattered waves possess the factor (*b/a*): each of them have to travel distance *a* either from 2 to 1 or from 1 to 2. For position 1, the incident wave has to travel an extra distance *a* from 1 to 2 and the scattered wave has to travel back the same distance → **the double phase delay**. For position 2, both the incident and the scattered wave have to travel the distance *a* from 1 to 2 → **same phase shift for both terms**.

(**Question 3B**)

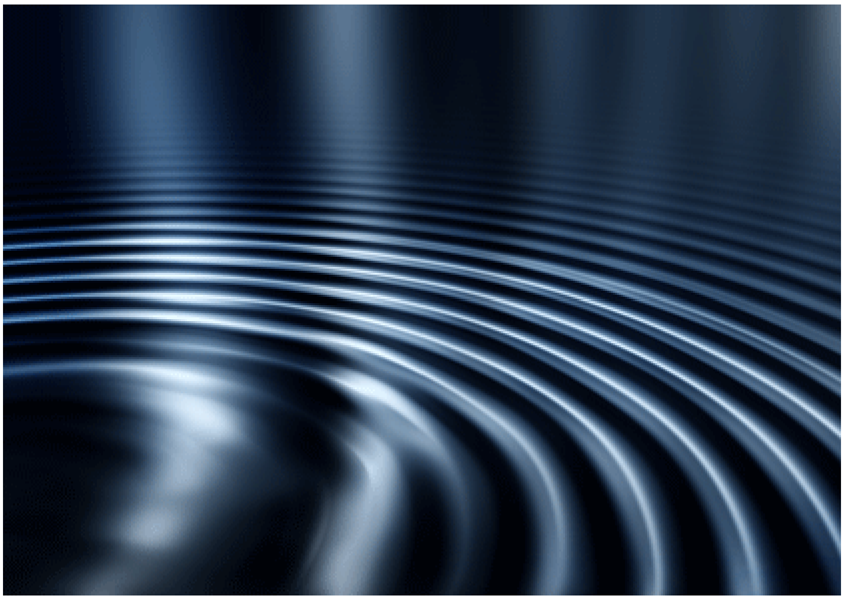
Calculate the intensity of the wave at the two positions.

After a simple calculation, similar to the one in (2B), one gets (assuming *b* is real)

$$I_1 = |A_0|^2 \left(1 + 2\frac{b}{a}\cos(ka) + \frac{b^2}{a^2}\right),$$

$$I_2 = |A_0|^2 \left(1 + 2\frac{b}{a} + \frac{b^2}{a^2}\right) = |A_0|^2 \left(1 + \frac{b}{a}\right)^2$$

Problem 4.



(**Question 4**)

Write down the expression for the amplitude $A(r,t)$ of a two-dimensional circular wave on the water surface. Neglect damping of the waves.

The energy conservation law requires that the total power taken away by the generated waves is the same at all distances if one neglects energy dissipation (damping) due to, e.g., viscosity. For a circular wave the energy is spread in 2D. The total power crossing through a circle of a radius *r* per unit time is proportional to $P \propto 2\pi r |A|^2$. Since it has to remain constant for all *r*, the amplitude of the circular wave has to be proportional to $1/\sqrt{r}$. Together with the phase factor, the wave can be written as

$$A(r,t) = \frac{A_0}{\sqrt{r}} e^{i\omega t - ikr}$$

I was surprised that only one student did it correct and earned a full point. A few students mentioned the energy conservation law (but no correct answer) and I gave them 0.25. The rest did not gain anything here. Well, this question was meant as "slightly advanced" and it made a distinction between a grade of 10 and the rest.