Write down your name clearly on every sheet of paper!
You can give your answers either in Dutch or English. Good luck!

Waves test, 26 November 2014

Mathematics:

Geometrical series:

$$1+x+x^2+x^3+\ldots+x^{N-1}=\frac{1-x^N}{1-x}$$

$$1+x+x^2+x^3+\ldots=\frac{1}{1-x}$$

$$1-x+x^2-x^3+\ldots=\frac{1}{1+x}$$

Taylor expansions:

$$\sqrt{1\pm x}\simeq 1\pm\frac{1}{2}x \qquad (x\ll 1)$$

$$\sqrt{A\pm x}\simeq\sqrt{A}\sqrt{1\pm\frac{x}{A}}\simeq\sqrt{A}\left(1\pm\frac{x}{2A}\right)=\sqrt{A}\pm\frac{x}{2\sqrt{A}} \qquad (x\ll A)$$

Trigonometric relations:

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha=1-2\sin^2\alpha=2\cos^2\alpha-1$$

$$\sin(2\alpha)=2\sin\alpha\cos\alpha$$

Complex exponent:

$$e^{\pm i\phi}=\cos\phi\pm i\sin\phi$$

$$e^{i\phi}e^{-i\phi}=1$$

Problem 1: using trigonometric functions

(**Question 1A**)
Using the trigonometric formulas on the first page, prove that
$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

Two waves are written as
$$A_1(x,t) = -A_0 \cos(\omega t + kx)$$
$$A_2(x,t) = A_0 \cos kx \cos \omega t$$

(**Question 1B**)
Which of the waves, $A_1(x,t)$ or $A_2(x,t)$, is a travelling wave? What is its phase velocity and in which direction the wave propagates?
Which of the waves, $A_1(x,t)$ or $A_2(x,t)$, is a standing wave? Calculate the positions of the nodes in this standing wave.

(**Question 1C**)
We now add the two waves together:
$$A_3(x,t) = A_1(x,t) + A_2(x,t)$$
What type of wave (travelling/standing/…) do we get as a result of their interference?
Hint: write the standing wave as a sum of two counter-propagating travelling waves.

Problem 2. using complex functions

Two waves with a period T and a wavelength λ are written as
$$A_1(x,t) = A_0 e^{i\omega t - ikx}$$
$$A_2(x,t) = A_0 e^{i\omega t + ikx + i\chi}$$

(Question 2A):
What type of the waves are they? Motivate your answer!
What is the relation between ω, k, T and λ?

(Question 2B):
Prove that the intensity of the interference pattern between the two waves $A_1(\vec{r},t) + A_2(\vec{r},t)$ can be written as
$$I(x) = 2|A_0|^2 (1 + \cos(2kx + \chi))$$

(Question 2C):
at what value of χ there will be a node at $x = 0$?

(Question 2C):
What is the distance between the nodes?

Problem 3.

Two identical atoms are located at $x = 0$ and $x = a$. They are irradiated by a plane wave travelling along the *x*-axis in the positive *x*-direction. Each atom can weakly scatter the wave and their scattering length is *b*.

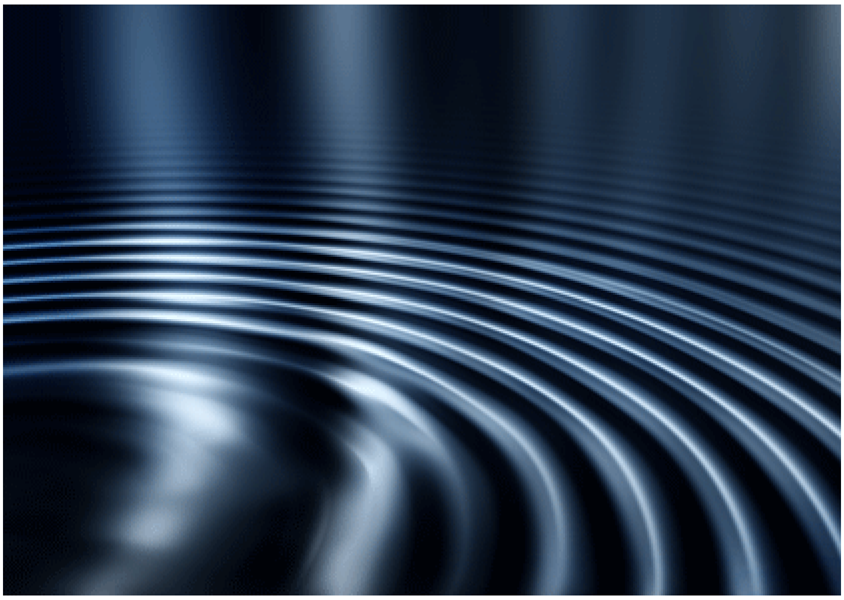
(**Question 3A**)
Make a sketch of the situation.
Argue that the total amplitude of the wave at the position of atoms 1 and 2 can be written as

$$A_1(x=0,t) = A_0 e^{i\omega t} + A_0 \frac{b}{a} e^{i\omega t - 2ika}$$

$$A_2(x=a,t) = A_0 e^{i\omega t - ika} + A_0 \frac{b}{a} e^{i\omega t - ika}$$

(**Question 3B**)
Calculate the intensity of the wave at the two positions.

Problem 4.



(**Question 4**)
Write down the expression for the amplitude $A(r,t)$ of a two-dimensional circular wave on the water surface. Neglect damping of the waves.