Sketch of suggested solutions

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Problem 1.

a) Observe that $2\cos(t) = e^{it} + e^{-it}$ and that $2 = e^0 + e^0 = e^{0\cdot t} + e^{0\cdot t}$ for $t \in \mathbb{R}$. Now use the superposition principle and the explicit form of a special solution to the inhomogeneous equation with an exponential function on the right-hand side to obtain the special solution

$$\begin{aligned} v_0(t) &= \frac{e^0}{(0)^2 + 3\cdot 0 + 2} + \frac{e^0}{(0)^2 + 3\cdot 0 + 2} - \frac{e^{it}}{(i)^2 + 3i + 2} - \frac{e^{-it}}{(-i)^2 - 3i + 2} \\ &= 1 - \frac{e^{it}}{1+3i} - \frac{e^{-it}}{1-3i}, \qquad t \in \mathbb{R}. \end{aligned}$$

Expressing the complex numbers $\frac{1}{1+3i} = \frac{1}{10}(1-3i)$ and $\frac{1}{1-3i} = \frac{1}{10}(1+3i)$ in polar coordinates, we find that

$$\frac{1}{1+3i} = \frac{1}{\sqrt{10}} e^{i\phi} \qquad \text{and} \qquad \frac{1}{1-3i} = \frac{1}{\sqrt{10}} e^{-i\phi}$$

with $\phi = -\arctan 3 \in (-\pi, \pi]$. Hence, $v_0(t) = 1 - \frac{1}{\sqrt{10}}(e^{i(t+\phi)} + e^{-i(t+\phi)}) = 1 - \frac{2}{\sqrt{10}}\cos(t+\phi)$, meaning that $a = 1$ and $b = -\frac{2}{\sqrt{10}}$.

b) Another solution to (1) can be obtained by adding to v_0 a non-trivial solution of the corresponding homogeneous equation, i.e.

$$f'' + 3f' + 2f = 0 \tag{5}$$

for $f : \mathbb{R} \to \mathbb{R}$. Consider the quadratic equation $\lambda^2 + 3\lambda + 2 = 0$ with discriminate $D = 1$ and solutions

$$\lambda_{1/2} = \frac{-3 \pm \sqrt{D}}{2} = \frac{1}{2}(-3 \pm 1).$$

Then $f(t) = e^{\lambda_1 t} = e^{-t}$ for $t \in \mathbb{R}$ is a solution to (5), and we find that v_1 given by $v_1(t) = v_0(t) + e^{-t}$ is a solution to (1), different from v_0.

c)* Since $\lim_{t\to\infty} e^{-t} = 0$, it holds that

$$\lim_{t\to\infty} |v_1(t) - v_0(t)| = \lim_{t\to\infty} e^{-t} = 0.$$

The difference between the solutions v_1 and v_0 converges to zero as t goes to infinity. In other words, v_1 and v_0 are arbitrarily close for sufficiently large t.

Problem 2.

a) The characteristic polynomial is

$$\det(A - \lambda\mathbb{I}) = \det\begin{pmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = \lambda^2(1-\lambda) - 4(1-\lambda) = (\lambda^2 - 4)(1-\lambda).$$

The eigenvalues are the roots of the characteristic polynomial, hence $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 1$.

An eigenvector $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{C}^3$ associated with $\lambda_1 = 2$ satisfies $Av = \lambda_1 v$, or equivalently solves the system of linear equations

$$\begin{aligned} 2v_2 - 2v_1 &= 0 \\ 2v_1 - 2v_2 &= 0 \\ -v_3 &= 0. \end{aligned}$$

Hence,

$$v = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{for some } c_1 \in \mathbb{C}.$$

Similarly, one obtains that

$$c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{with } c_2 \in \mathbb{C}, \qquad \text{and} \quad c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{with } c_3 \in \mathbb{C},$$

are the eigenvectors corresponding to $\lambda_2 = -2$ and $\lambda_3 = 1$, respectively.

b) Since A is diagonalizable, the general solution to $\frac{d}{dt}F = AF$ is given by

$$F(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for $t \in \mathbb{R}$ with $c_1, c_2, c_3 \in \mathbb{C}$.

c) In order to find the solution that satisfies $F(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ we choose the constants c_1, c_2 and c_3 such that

$$F(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, $c_2 = c_1 = \frac{1}{2}$ and $c_3 = 1$.

Problem 3.

a) By differentiating every term individually one finds that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

b) Calculating that

$$\begin{aligned} (x^2 - x) f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n, \\ (2x - 1) f'(x) &= \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \\ \tfrac{1}{4} f(x) &= \sum_{n=0}^{\infty} \tfrac{1}{4} a_n x^n, \end{aligned}$$

we infer from the identity principle by equating coefficients of the same order that

$$\begin{aligned} \tfrac{1}{4}a_0 - a_1 &= 0 \\ \tfrac{9}{4}a_1 - 4a_2 &= 0 \\ (n^2+n+\tfrac{1}{4})a_n - (n+1)^2 a_{n+1} &= 0 \qquad \text{for } n \geq 2. \end{aligned}$$

Hence,

$$a_{n+1} = \frac{(n+\frac{1}{2})^2}{(n+1)^2} a_n = \left(\frac{n+\frac{1}{2}}{n+1}\right)^2 a_n$$

for $n \geq 0$.

c) In view of $a_0 = 1$ and the fact that $|\frac{n+\frac{1}{2}}{n+1}| < 1$ for all $n \geq 0$, it follows that

$$|a_n| < |a_{n-1}| < \ldots < |a_1| < |a_0| = 1 \tag{6}$$

for all $n \geq 1$. Since we know that the geometric series $\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$, we obtain from the comparison test in view of (6) that $\sum_{n=0}^{\infty} a_n x^n$ converges for $x \in (-1,1)$.

Problem 4*.

Remark: Instead of $L(f)$ we often write Lf.

a) Let $f, g \in C^\infty(\mathbb{R})$. Then

$$\begin{aligned} (L(f+g))(x) &= (f+g)''(x) + 2(f+g)'(x) = f''(x) + g''(x) + 2f'(x) + 2g'(x) \\ &= f''(x) + 2f'(x) + g''(x) + 2g'(x) = (Lf)(x) + (Lg)(x) \end{aligned}$$

for all $x \in \mathbb{R}$. This shows that $L(f+g) = Lf + Lg$.

Let $f \in C^\infty(\mathbb{R})$ and $r \in \mathbb{R}$. Then

$$(L(rf))(x) = (rf)''(x) + 2(rf)'(x) = rf''(x) + 2rf'(x) = r(f''(x) + 2f'(x)) = r(Lf)(x)$$

for all $x \in \mathbb{R}$. This shows that $L(rf) = r(Lf)$.

Hence, L is a linear operator.

b) We know that $\lambda = -1$ is an eigenvalue of L if and only if the differential equation

$$f'' + 2f' + f = 0 \tag{7}$$

for $f : \mathbb{R} \to \mathbb{R}$ has a nontrivial solution. To find the general solution to (7) consider the polynomial $\mu^2 + 2\mu + 1 = 0$ with root $\mu = -1$. Hence,

$$f(x) = c_1 e^{-x} + c_2 x e^{-x}, \qquad x \in \mathbb{R},$$

with constants $c_1, c_2 \in \mathbb{R}$ is the general solution to (7).

Consequently, $\lambda = -1$ is an eigenvalue of L and the corresponding eigenspace is

$$E_{-1} = \{f \in C^\infty(\mathbb{R}) : f(x) = c_1 e^{-x} + c_2 x e^{-x}, c_1, c_2 \in \mathbb{R}\} = \operatorname{span}(e^{-x}, xe^{-x}).$$

c) Since there are nontrivial solutions to the differential equation $f'' + 2f' - \lambda f = 0$ in $\mathbb{R}$ for every $\lambda \in \mathbb{R}$, the spectrum of L consists of all real numbers.