Sketch of suggested solutions

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Problem 1.

a) The length of v is $\|v\| = \sqrt{\langle v, v\rangle} = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{1+1+0} = \sqrt{2}$ and similarly $\|w\| = \sqrt{2}$. Moreover,

$$\langle v, w\rangle = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 1.$$

b) We show that Z is non-empty and closed under addition and scalar multiplication:

- Choosing $a = b = 0$ shows that $0 \in Z$.
- Let $u, z \in Z$, then $z = av + bw$ and $u = cv + dw$ for $a, b, c, d \in \mathbb{R}$ and

  $$z + u = av + bw + cv + dw = (a + c)v + (b + d)w.$$

  Since $a + b, c + d \in \mathbb{R}$, we find that $z + u \in Z$.
- Let $z \in Z$ and $r \in \mathbb{R}$. Then $z = av + bw$ with $a, b \in \mathbb{R}$ and $rz = r(av + bw) = (ra)v + (rb)w$. As $ra, rb \in \mathbb{R}$, it follows that $rz \in Z$.

c) Following the Gram-Schmidt orthonormalization procedure, we set

$$v^* = \frac{v}{\|v\|} \overset{a)}{=} \frac{1}{\sqrt{2}} v,$$

and

$$\tilde{w} = w - \langle w, v^*\rangle v^* = w - \frac{\langle w, v\rangle}{\|v\|^2} v = w - \frac{1}{2} v = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

One can easily double-check that v^* and $\tilde{w}$ are indeed orthogonal. It remains to normalize $\tilde{w}$. As

$$\|\tilde{w}\|^2 = \tilde{w}_1^2 + \tilde{w}_2^2 + \tilde{w}_3^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2 = \frac{3}{2},$$

we set

$$w^* = \frac{\tilde{w}}{\|\tilde{w}\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

By construction, the vectors v^* and w^* form an orthonormal basis of Z.

d) Let $z = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$. One obtains for the projection $\mathcal{P}z$ of z onto Z that

$$\mathcal{P}z = \langle z, v^*\rangle v^* + \langle z, w^*\rangle w^* = \frac{1}{2}\langle z, v\rangle v + \frac{4}{6}\langle z, \tilde{w}\rangle \tilde{w} = 3v + 2\tilde{w} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}.$$

Problem 2.

a) Let $p, q \in \mathcal{P}_{\mathbb{R}}$ and $r, s \in \mathbb{R}$, then by the sum rule of differentiation,

$$\begin{aligned}(D(rp+sq))(x) &= x^2(rp+sq)''(x) + 2x(rp+sq)'(x) = x^2(rp''(x)+sq''(x)) + 2x(rp'(x)+sq'(x)) \\ &= r(x^2p''(x)+2xp'(x)) + s(x^2q''(x)+2xq'(x)) = r(Dp)(x) + s(Dq)(x)\end{aligned}$$

for all $x \in \mathbb{R}$. This shows that D is a linear operator.

b) Let $k \in \mathbb{N}_0$. Then for $x \in \mathbb{R}$,

$$(Dp_k)(x) = x^2k(k-1)x^{k-2} + 2xkx^{k-1} = k(k-1)x^k + 2kx^k = (k^2-k+2k)x^k = (k^2+k)p_k(x).$$

Hence, $Dp_k = (k^2+k)p_k$. Since p_k is non-trivial, this shows that p_k is an eigenfunction of D and the corresponding eigenvalue is k^2+k.

c) To show that $\langle \cdot, \cdot \rangle$ is an inner product, we prove linearity in the first argument, symmetry, positivity and definiteness:

- Linearity in the first argument and symmetry are immediate consequences of the linearity of the integral and the properties of $\mathbb{R}$. More precisely, for $p, \tilde{p}, q \in \mathcal{P}_{\mathbb{R}}$ and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\langle \alpha p + \beta\tilde{p}, q\rangle &= \int_{-\infty}^{\infty} (\alpha p(x) + \beta\tilde{p}(x))q(x)e^{-x^2}\,dx \\ &= \alpha\int_{-\infty}^{\infty} p(x)q(x)e^{-x^2}\,dx + \beta\int_{-\infty}^{\infty} \tilde{p}(x)q(x)e^{-x^2}\,dx = \alpha\langle p, q\rangle + \beta\langle \tilde{p}, q\rangle.\end{aligned}$$

  Similarly one can prove that $\langle p, q\rangle = \langle q, p\rangle$.
- For all $p \in \mathcal{P}_{\mathbb{R}}$ it holds that $\langle p, p\rangle = \int_{-\infty}^{\infty} p(x)^2e^{-x^2}\,dx \geq 0$, since the integrand $p(x)^2e^{-x^2}$ is non-negative for every $x \in \mathbb{R}$.
- If $0 = \langle p, p\rangle = \int_{\infty}^{\infty} p(x)^2e^{-x^2}\,dx$, then the hint tells us that $p(x)^2e^{-x^2} = 0$ for all $x \in \mathbb{R}$. Since $e^{-x^2} > 0$, it follows that $p(x)^2 = 0$ and thus $p(x) = 0$ for all $x \in \mathbb{R}$, meaning that p is the null polynomial.

d) Accounting for b) gives that $Dp_1 = 2p_1$. Then

$$\langle Dp_1, p_0\rangle = 2\langle p_1, p_0\rangle = \int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \lim_{N\to-\infty} e^{-N^2} - \lim_{N\to\infty} e^{-N^2} = 0.$$

This calculation shows that Dp_1 and p_0 are orthogonal.

Problem 3.

a) The visualization of f is left to the reader. We observe that f is piecewise continuously differentiable.

b) For $k = 0$ we have that

$$\hat{f}_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \int_0^1 -3x+1\,dx = -\frac{3}{2} + 1 = -\frac{1}{2}.$$

In the case $k \neq 0$, we use integration by parts to obtain

$$\begin{aligned}
\hat{f}_k &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-ik\pi x}\,dx = \underbrace{\frac{1}{2}\int_{-1}^{1} e^{-ik\pi x}\,dx}_{=0} + \frac{3}{2}\Big[\int_{-1}^{0} xe^{-ik\pi x}\,dx - \int_{0}^{1} xe^{-ik\pi x}\,dx\Big] \\
&= \frac{3}{2}\Big[\int_{-1}^{0} x\cos(k\pi x)\,dx - \int_{0}^{1} x\cos(k\pi x)\,dx \underbrace{- \int_{-1}^{0} x\sin(k\pi x)\,dx + \int_{0}^{1} x\sin(k\pi x)\,dx\Big]}_{=0} \\
&= 3\int_{-1}^{0} x\cos(k\pi x)\,dx = -3\int_{-1}^{0} \frac{1}{k\pi}\sin(k\pi x)\,dx = \frac{3}{k^2\pi^2}(1-\cos(k\pi x)) \\
&= \begin{cases} 0 & \text{if } k \text{ even,} \\ \frac{6}{k^2\pi^2} & \text{if } k \text{ odd.} \end{cases}
\end{aligned}$$

c) Since f is 2-periodic, piecewise continuously differentiable and continuous, the Fourier inversion formula tells us that for $x \in \mathbb{R}$,

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\pi x}\,dx,$$

where the series on the right-hand side converges. As a consequence of the Euler formula we can write

$$\begin{aligned}
f(x) &= \hat{f}_0 + \sum_{k=1}^{\infty} (\hat{f}_k + \hat{f}_{-k})\cos(k\pi x) + i(\hat{f}_k - \hat{f}_{-k})\sin(k\pi x) \\
&= a_0 + \sum_{k=1}^{\infty} a_k\cos(k\pi x) + b_k\sin(k\pi x), \qquad x \in \mathbb{R},
\end{aligned}$$

with $a_0 = \hat{f}_0$, $a_k = \hat{f}_k + \hat{f}_{-k}$ and $b_k = i(\hat{f}_k - \hat{f}_{-k})$ for $k \in \mathbb{N}$. Since f is even, we have that $\hat{f}_k = \hat{f}_{-k}$ for all $k \in \mathbb{Z}$. Consequently, $b_k = 0$ and $a_k = 2\hat{f}_k$ for $k \in \mathbb{N}$, so that

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\pi x) = \sum_{k=0}^{\infty} a_k\cos(k\pi x), \qquad x \in \mathbb{R}. \tag{5}$$

Using the calculations in b), we have that $a_0 = -\frac{1}{2}$, and $a_k = 0$ for k even and $a_k = \frac{12}{k^2\pi^2}$ for k odd.

d) Setting $x = 0$ in (5) results in

$$1 = f(0) = -\frac{1}{2} + \sum_{k=1,\,k\,\mathrm{odd}} \frac{12}{k^2\pi^2},$$

or equivalently,

$$\frac{\pi^2}{8} = \sum_{k=1,\,k\,\mathrm{odd}} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots.$$

Problem 4.

a) Observe that with $\mathcal{F}$ denoting the Fourier transformation with respect to the x-variable, we obtain for every $t > 0$ that

$$\mathcal{F}(t\tfrac{\partial}{\partial x}u(\cdot,t))(s) = t(is)\hat{u}(s,t) = ist\,\hat{u}(s,t), \qquad s \in \mathbb{R},$$

and

$$\mathcal{F}(\tfrac{\partial}{\partial t}u(\cdot,t))(s) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}u(x,t)e^{-isx}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-isx}\,dx = \frac{\partial}{\partial t}\hat{u}(s,t), \qquad s \in \mathbb{R}.$$

Hence, applying $\mathcal{F}$ to (1) and (2) results in

$$\frac{\partial}{\partial t}\hat{u}(s,t) + ist\,\hat{u}(s,t) = 0 \qquad \text{and} \qquad \hat{u}(s,0) = \hat{g}(s)$$

for every $s \in \mathbb{R}$, or in other words, $\hat{u}(s,\cdot)$ solves the first order differential equation with non-constant coefficients

$$v'(t) + ist\,v(t) = 0, \qquad t > 0, \tag{6}$$

subject to the initial condition $v(0) = \hat{g}(s)$.

b)We verify that the suggested formula solves (6) by plugging it in. Indeed, for every $s \in \mathbb{R}$ we find with the chain rule that $\frac{d}{dt}e^{-i\frac{t^2}{2}s} = -ist\,e^{-i\frac{t^2}{2}s}$, so that

$$\frac{d}{dt}(\hat{g}(s)e^{-i\frac{t^2}{2}s}) + ist\,\hat{g}(s)e^{-i\frac{t^2}{2}s} = 0.$$

Moreover, $\hat{u}(s,0) = \hat{g}(s)e^{-i\frac{0^2}{2}\cdot s} = \hat{g}(s)$. This shows that $\hat{u}(s,\cdot)$ solves the initial value problem derived in b).

c) Since $\hat{g}(0) = \int_{-\infty}^{\infty} g(x)\,dx$ and $\hat{u}(0,t) = \int_{-\infty}^{\infty} u(x,t)\,dx$ for every $t > 0$, we find in view of b) that

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \hat{u}(0,t) = \hat{g}(0)e^{-i\frac{t^2}{2}\cdot 0} = \hat{g}(0) = \int_{-\infty}^{\infty} g(x)\,dx.$$

Consequently, if $\int_{-\infty}^{\infty} g(x)\,dx = 0$, then $\int_{-\infty}^{\infty} u(x,t)\,dx = 0$ for all $t > 0$.

d) It follows from the change of variables $y = x - \alpha$ that

$$\begin{aligned}\hat{f}_\alpha(s) &= \int_{-\infty}^{\infty} f_\alpha(x)e^{-isx}\,dx = \int_{-\infty}^{\infty} f(x-\alpha)e^{-isx}\,dx = \int_{-\infty}^{\infty} f(y)e^{-is(y+\alpha)}\,dy \\ &= e^{-is\alpha}\int_{-\infty}^{\infty} f(y)e^{-isy}\,dy = e^{-is\alpha}\hat{f}(s), \qquad s \in \mathbb{R}.\end{aligned}$$

e) In view of b) and d) one obtains that

$$\hat{u}(s,t) = \hat{g}(s)e^{-i\frac{t^2}{2}s} = \hat{g}_{\frac{t^2}{2}}(s), \qquad s \in \mathbb{R},\ t \geq 0.$$

Using Fourier inversion we infer for every $t \geq 0$ that

$$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\cdot,t))(x) = g_{\frac{t^2}{2}}(x) = g\Big(x - \frac{t^2}{2}\Big), \qquad x \in \mathbb{R}.$$

Problem 5.

a) The visualization of f is left to the reader. We observe that f is piecewise continuously differentiable and has compact support, in particular is f absolutely integrable.

b) If $s \neq 0$ we find that

$$\begin{aligned}\hat{f}(s) &= \int_{-\infty}^{\infty} f(t)e^{-ist}\,dt = \int_{-1}^{1} e^{-ist}\,dt - \int_{-1}^{1} t^2 e^{-ist}\,dt\\ &= \frac{2\sin(s)}{s} + \frac{4\sin(s)}{s^3} - \frac{4\cos(s)}{s^2} - \frac{2\sin(s)}{s} = 4g(s).\end{aligned}$$

In the third equality we have used the result of the following computation, for which we apply integration by parts twice,

$$\begin{aligned}\int_{-1}^{1} t^2 e^{-its}\,dt &= \frac{2}{is}\int_{-1}^{1} te^{-ist}\,dt + \frac{1}{is}(e^{is} - e^{-is})\\ &= \frac{2}{(is)^2}\int_{-1}^{1} e^{-ist}\,dt - \frac{2}{(is)^2}(e^{is} + e^{-is}) + 2\frac{\sin(s)}{s}\\ &= \frac{2}{-s^2}\Big(\frac{1}{-is}e^{-is} + \frac{1}{is}e^{is}\Big) + \frac{4\cos(s)}{s^2} + \frac{2\sin(s)}{s} = -\frac{4\sin(s)}{s^3} + \frac{4\cos(s)}{s^2} + \frac{2\sin(s)}{s}.\end{aligned}$$

In the case $s = 0$ one has that

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(t)\,dt = 2\int_{0}^{1} 1 - t^2\,dt = 2\Big(1 - \frac{1}{3}\Big) = \frac{4}{3}.$$

Finally, since $\hat{f}$ is continuous, we see that $\hat{f} = ag$ with $a = 4$.

c) From b) and Plancherel's formula we infer that

$$\begin{aligned}\int_{-\infty}^{\infty} |g(s)|^2\,ds &= \frac{1}{4^2}\int_{-\infty}^{\infty} |\hat{f}(s)|^2\,ds = \frac{\pi}{8}\int_{-\infty}^{\infty} |f(s)|^2\,ds = \frac{\pi}{8}\int_{-1}^{1}(1 - t^2)^2\,dt\\ &= \frac{\pi}{4}\int_{0}^{1} 1 - 2t^2 + t^4\,dt = \frac{2\pi}{15}.\end{aligned}$$

Problem 6*.

a) Let $v(t) = \begin{pmatrix} f(t)\\ f'(t)\end{pmatrix}$ for all $t \in \mathbb{R}$. Then

$$v' = \begin{pmatrix} f'\\ f''\end{pmatrix} = \begin{pmatrix} 0 & 1\\ -\frac{k}{m} & 0\end{pmatrix}\begin{pmatrix} f\\ f'\end{pmatrix} = Av \qquad \text{with } A = \begin{pmatrix} 0 & 1\\ -\dfrac{k}{m} & 0\end{pmatrix}.$$

b) Solving (4) with initial value $v(0) = e_1$ is equivalent to solving (3) subject to the initial conditions $f(0) = 1$ and $f'(0) = 0$. Since the roots of the polynomial $\lambda^2 + \frac{k}{m}$ are $\lambda_{1/2} = \pm i\sqrt{\frac{k}{m}}$, the general solution to (3) is $f(t) = \alpha e^{i\sqrt{\frac{k}{m}}t} + \beta e^{-i\sqrt{\frac{k}{m}}t}$ with $\alpha, \beta \in \mathbb{C}$. We infer from the initial conditions that

$$1 = f(0) = \alpha + \beta \qquad \text{and} \qquad 0 = f'(0) = i\sqrt{\frac{k}{m}}(\alpha - \beta).$$

Hence $\alpha = \beta = \frac{1}{2}$ and $f(t) = \cos\big(\sqrt{\frac{k}{m}}t\big)$ for $t \in \mathbb{R}$. Consequently, the sought solution is

$$v(t) = \begin{pmatrix} \cos\big(\sqrt{\frac{k}{m}}t\big) \\ -\sqrt{\frac{k}{m}}\sin\big(\sqrt{\frac{k}{m}}t\big) \end{pmatrix}, \qquad t \in \mathbb{R}.$$

c) For all $v, w \in \mathbb{C}^2$ and $\alpha, \beta \in \mathbb{C}$ one obtains that

$$H(\alpha v + \beta w) = iA(\alpha v + \beta w) = iA(\alpha v) + iA(\beta w) = \alpha iAv + \beta iAw = \alpha H(v) + \beta H(w),$$

which shows that H is a linear operator.

d) We will prove that H is Hermitian if and only if $k = m$. Observing that $Av = \begin{pmatrix} v_2 \\ -\frac{k}{m}v_1 \end{pmatrix}$ for $v \in \mathbb{C}^2$, we obtain for all $v, w \in \mathbb{C}^2$ that

$$\begin{aligned}\langle Hv, w\rangle - \langle v, Hw\rangle &= i\langle Av, w\rangle - (-i)\langle v, Aw\rangle = i(\langle Av, w\rangle + \langle v, Aw\rangle) \\ &= i\big(v_2\overline{w}_1 - \tfrac{k}{m}v_1\overline{w}_2 + v_1\overline{w}_2 - \tfrac{k}{m}v_2\overline{w}_1\big) = i(1 - \tfrac{k}{m})(v_1\overline{w}_2 + v_2\overline{w}_1).\end{aligned}$$

Since in general $v_1\overline{w}_2 + v_2\overline{w}_1 \neq 0$ (take e.g. $v = e_1$ and $w = e_2$), this implies that H is Hermitian if and only if $1 - \frac{k}{m} = 0$, or equivalently $k = m$.

e) Considering that v is a solution to (4) we find that

$$\begin{aligned}\frac{d}{dt}\|v(t)\|^2 &= \frac{d}{dt}\langle v(t), v(t)\rangle = \langle v'(t), v(t)\rangle + \langle v(t), v'(t)\rangle = \langle Av(t), v(t)\rangle + \langle v(t), Av(t)\rangle = \\ &= \langle -iHv(t), v(t)\rangle + \langle v(t), -iHv(t)\rangle = -i\langle Hv(t), v(t)\rangle + i\langle v(t), Hv(t)\rangle = 0, \qquad t \in \mathbb{R}.\end{aligned}$$

In the last step we used the fact that H is Hermitian.

As a consequence, the norm of $v(t)$ is constant for all $t \in \mathbb{R}$. In the case $k = m$ the solution from b) lives on a circle in the two-dimensional real plane.