

## Midterm exam

**Date:** Monday, March 6, 2017

**Time:** 09:15 - 10:45 (90 min)

**Room:** Educatorium, Room BETA

**Instructions:**

- Write your *name, student number, and problem number* on every page you hand in.
  - The use of textbooks, notes, calculators, and cell phones is *not* allowed.
  - Justify your answers.
  - Problem 4\* and Problem 1c)\* are bonus questions.
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**Problem 1.**

Consider the inhomogeneous differential equation

$$v'' + 3v' + 2v = g \tag{1}$$

for  $v = v(t) : \mathbb{R} \rightarrow \mathbb{R}$ , where  $g : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $g(t) = 2 - 2\cos(t)$  for  $t \in \mathbb{R}$ .

a) Determine the special solution  $v_0$  to (1) that can be written in the form

$$v_0(t) = a + b \cos(t + \phi), \quad t \in \mathbb{R},$$

with  $a, b \in \mathbb{R}$  and  $\phi \in (-\pi, \pi]$ . Give explicit expressions for  $a$  and  $b$ . 6p

b) Find another solution  $v_1$  to the differential equation (1) that is different from  $v_0$  in a). 3p

c)\* Consider  $\lim_{t \rightarrow \infty} |v_1(t) - v_0(t)|$  with  $v_0$  as in a) and  $v_1$  as in b). What do you observe? 1p

**Problem 2.**

a) Determine the eigenvalues and eigenvectors of the  $3 \times 3$  matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

6p

b) Give the general solution to the vectorial differential equation

$$\frac{d}{dt}F = AF \tag{2}$$

for  $F = F(t) : \mathbb{R} \rightarrow \mathbb{C}^3$ , where  $A$  is the matrix introduced in a). *Hint:* Note that  $A$  is symmetric and therefore diagonalizable. You may use this observation without further proof. 2p

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- c) Which of the solutions to (2) found in b) meets the initial condition  $F(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ? 2p

### Problem 3.

We aim at finding a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (3)$$

with real coefficients  $a_n$  that satisfies

$$(x^2 - x)f''(x) + (2x - 1)f'(x) + \frac{1}{4}f(x) = 0 \quad (4)$$

for all  $x$  in a suitable real interval.

- a) Determine the first and second derivative  $f'$  and  $f''$  of the power series  $f$  defined by (3). 2p
- b) Assuming that  $f$  as in (3) solves the differential equation (4), verify that the following recurrence relation for the coefficients  $a_n$  holds:

$$(n+1)^2 a_{n+1} = (n + \frac{1}{2})^2 a_n \quad \text{for all integers } n \geq 0.$$

*Hint:* Treat the zeroth and first order terms separately. 6p

- c) Let  $a_0 = 1$ . Show that then  $|a_n| < 1$  for all integers  $n \geq 1$ , where  $a_n$  are the coefficients found in b). Use this observation to prove that the power series  $\sum_{n=0}^{\infty} a_n x^n$  converges for all  $x \in (-1, 1)$ . 3p

### Problem 4\*.

Let  $L : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$  be defined by  $L(f) = f'' + 2f'$  for  $f \in C^\infty(\mathbb{R})$ .

- a) Show that  $L$  is a linear operator on the linear space  $C^\infty(\mathbb{R})$  over  $\mathbb{R}$ . 3p
- b) Is  $\lambda = -1$  an eigenvalue of  $L$ ? If so, determine the corresponding eigenspace. 4p
- c) What is the spectrum of  $L$ ? *Hint:* It is enough to give a brief argument to support your statement, a full proof is not required. 2p

## Sketch of suggested solutions

Please email errors and/or suggestions to c.kreisbeck@uu.nl.

### Problem 1.

a) Observe that  $2 \cos(t) = e^{it} + e^{-it}$  and that  $2 = e^0 + e^0 = e^{0 \cdot t} + e^{0 \cdot t}$  for  $t \in \mathbb{R}$ . Now use the superposition principle and the explicit form of a special solution to the inhomogeneous equation with an exponential function on the right-hand side to obtain the special solution

$$\begin{aligned} v_0(t) &= \frac{e^0}{(0)^2 + 3 \cdot 0 + 2} + \frac{e^0}{(0)^2 + 3 \cdot 0 + 2} - \frac{e^{it}}{(i)^2 + 3i + 2} - \frac{e^{-it}}{(-i)^2 - 3i + 2} \\ &= 1 - \frac{e^{it}}{1 + 3i} - \frac{e^{-it}}{1 - 3i}, \quad t \in \mathbb{R}. \end{aligned}$$

Expressing the complex numbers  $\frac{1}{1+3i} = \frac{1}{10}(1-3i)$  and  $\frac{1}{1-3i} = \frac{1}{10}(1+3i)$  in polar coordinates, we find that

$$\frac{1}{1+3i} = \frac{1}{\sqrt{10}}e^{i\phi} \quad \text{and} \quad \frac{1}{1-3i} = \frac{1}{\sqrt{10}}e^{-i\phi}$$

with  $\phi = -\arctan 3 \in (-\pi, \pi]$ . Hence,  $v_0(t) = 1 - \frac{1}{\sqrt{10}}(e^{i(t+\phi)} + e^{-i(t+\phi)}) = 1 - \frac{2}{\sqrt{10}} \cos(t + \phi)$ , meaning that  $a = 1$  and  $b = -\frac{2}{\sqrt{10}}$ .

b) Another solution to (1) can be obtained by adding to  $v_0$  a non-trivial solution of the corresponding homogeneous equation, i.e.

$$f'' + 3f' + 2f = 0 \tag{5}$$

for  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Consider the quadratic equation  $\lambda^2 + 3\lambda + 2 = 0$  with discriminant  $D = 1$  and solutions

$$\lambda_{1/2} = \frac{-3 \pm \sqrt{D}}{2} = \frac{1}{2}(-3 \pm 1).$$

Then  $f(t) = e^{\lambda_1 t} = e^{-t}$  for  $t \in \mathbb{R}$  is a solution to (5), and we find that  $v_1$  given by  $v_1(t) = v_0(t) + e^{-t}$  is a solution to (1), different from  $v_0$ .

c)\* Since  $\lim_{t \rightarrow \infty} e^{-t} = 0$ , it holds that

$$\lim_{t \rightarrow \infty} |v_1(t) - v_0(t)| = \lim_{t \rightarrow \infty} e^{-t} = 0.$$

The difference between the solutions  $v_1$  and  $v_0$  converges to zero as  $t$  goes to infinity. In other words,  $v_1$  and  $v_0$  are arbitrarily close for sufficiently large  $t$ .

### Problem 2.

a) The characteristic polynomial is

$$\det(A - \lambda \mathbb{I}) = \det \begin{pmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = \lambda^2(1 - \lambda) - 4(1 - \lambda) = (\lambda^2 - 4)(1 - \lambda).$$

The eigenvalues are the roots of the characteristic polynomial, hence  $\lambda_1 = 2$ ,  $\lambda_2 = -2$ ,  $\lambda_3 = 1$ .

An eigenvector  $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{C}^3$  associated with  $\lambda_1 = 2$  satisfies  $Av = \lambda_1 v$ , or equivalently solves the system of linear equations

$$\begin{aligned} 2v_2 - 2v_1 &= 0 \\ 2v_1 - 2v_2 &= 0 \\ -v_3 &= 0. \end{aligned}$$

Hence,

$$v = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for some } c_1 \in \mathbb{C}.$$

Similarly, one obtains that

$$c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{with } c_2 \in \mathbb{C}, \quad \text{and} \quad c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{with } c_3 \in \mathbb{C},$$

are the eigenvectors corresponding to  $\lambda_2 = -2$  and  $\lambda_3 = 1$ , respectively.

b) Since  $A$  is diagonalizable, the general solution to  $\frac{d}{dt}F = AF$  is given by

$$F(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for  $t \in \mathbb{R}$  with  $c_1, c_2, c_3 \in \mathbb{C}$ .

c) In order to find the solution that satisfies  $F(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  we choose the constants  $c_1, c_2$  and  $c_3$  such that

$$F(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence,  $c_2 = c_1 = \frac{1}{2}$  and  $c_3 = 1$ .

### Problem 3.

a) By differentiating every term individually one finds that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

b) Calculating that

$$\begin{aligned} (x^2 - x)f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n, \\ (2x - 1)f'(x) &= \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \\ \frac{1}{4}f(x) &= \sum_{n=0}^{\infty} \frac{1}{4} a_n x^n, \end{aligned}$$

we infer from the identity principle by equating coefficients of the same order that

$$\begin{aligned}\frac{1}{4}a_0 - a_1 &= 0 \\ \frac{9}{4}a_1 - 4a_2 &= 0 \\ (n^2 + n + \frac{1}{4})a_n - (n+1)^2a_{n+1} &= 0 \quad \text{for } n \geq 2.\end{aligned}$$

Hence,

$$a_{n+1} = \frac{(n + \frac{1}{2})^2}{(n+1)^2} a_n = \left(\frac{n + \frac{1}{2}}{n+1}\right)^2 a_n$$

for  $n \geq 0$ .

c) In view of  $a_0 = 1$  and the fact that  $|\frac{n+\frac{1}{2}}{n+1}| < 1$  for all  $n \geq 0$ , it follows that

$$|a_n| < |a_{n-1}| < \dots < |a_1| < |a_0| = 1 \quad (6)$$

for all  $n \geq 1$ . Since we know that the geometric series  $\sum_{n=0}^{\infty} x^n$  converges for  $|x| < 1$ , we obtain from the comparison test in view of (6) that  $\sum_{n=0}^{\infty} a_n x^n$  converges for  $x \in (-1, 1)$ .

#### Problem 4\*.

*Remark:* Instead of  $L(f)$  we often write  $Lf$ .

a) Let  $f, g \in C^\infty(\mathbb{R})$ . Then

$$\begin{aligned}(L(f+g))(x) &= (f+g)''(x) + 2(f+g)'(x) = f''(x) + g''(x) + 2f'(x) + 2g'(x) \\ &= f''(x) + 2f'(x) + g''(x) + 2g'(x) = (Lf)(x) + (Lg)(x)\end{aligned}$$

for all  $x \in \mathbb{R}$ . This shows that  $L(f+g) = Lf + Lg$ .

Let  $f \in C^\infty(\mathbb{R})$  and  $r \in \mathbb{R}$ . Then

$$(L(rf))(x) = (rf)''(x) + 2(rf)'(x) = rf''(x) + 2rf'(x) = r(f''(x) + 2f'(x)) = r(Lf)(x)$$

for all  $x \in \mathbb{R}$ . This shows that  $L(rf) = r(Lf)$ .

Hence,  $L$  is a linear operator.

b) We know that  $\lambda = -1$  is an eigenvalue of  $L$  if and only if the differential equation

$$f'' + 2f' + f = 0 \quad (7)$$

for  $f : \mathbb{R} \rightarrow \mathbb{R}$  has a nontrivial solution. To find the general solution to (7) consider the polynomial  $\mu^2 + 2\mu + 1 = 0$  with root  $\mu = -1$ . Hence,

$$f(x) = c_1 e^{-x} + c_2 x e^{-x}, \quad x \in \mathbb{R},$$

with constants  $c_1, c_2 \in \mathbb{R}$  is the general solution to (7).

Consequently,  $\lambda = -1$  is an eigenvalue of  $L$  and the corresponding eigenspace is

$$E_{-1} = \{f \in C^\infty(\mathbb{R}) : f(x) = c_1 e^{-x} + c_2 x e^{-x}, c_1, c_2 \in \mathbb{R}\} = \text{span}(e^{-x}, x e^{-x}).$$

c) Since there are nontrivial solutions to the differential equation  $f'' + 2f' - \lambda f = 0$  in  $\mathbb{R}$  for every  $\lambda \in \mathbb{R}$ , the spectrum of  $L$  consists of all real numbers.