

Final exam

Date: Monday, April 10, 2017
Time: 09:00 - 11:30 (150 min)
Room: Educatorium, Room BETA

1	2	3	4	5	6	Σ

Name:

Instructions:

- Write your *name, student number, and problem number* on every page you hand in.
- The use of textbooks, notes, calculators, and cell phones is *not* allowed.
- Justify your answers.
- Problem 6* and Problem 4 d)* are bonus questions.

Problem 1.

In \mathbb{R}^3 with the standard inner product we consider the vectors

$$v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

a) Determine the length of the vectors v and w , as well as the inner product $\langle v, w \rangle$. 2p

b) Show that $Z = \{av + bw : a, b \in \mathbb{R}\}$ is a linear subspace of \mathbb{R}^3 . 2p

c) The vectors v and w form a basis of Z , but they are neither orthogonal nor normalized, cf. a).
Apply the Gram-Schmidt algorithm to v and w to find an orthonormal basis of Z . 3p

d) Determine the orthogonal projection of $z = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ onto the linear subspace Z . 2p

Problem 2.

Let $\mathcal{P}_{\mathbb{R}}$ be the linear space (over \mathbb{R}) of all polynomials with real coefficients. We define the map $D : \mathcal{P}_{\mathbb{R}} \rightarrow \mathcal{P}_{\mathbb{R}}$ by

$$(Dp)(x) = x^2 p''(x) + 2xp'(x), \quad x \in \mathbb{R},$$

for every $p \in \mathcal{P}_{\mathbb{R}}$.

a) Prove that D is a linear operator. 2p

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b) Verify the following statement: For every integer $k \geq 0$ the polynomial p_k with $p_k(x) = x^k$ for $x \in \mathbb{R}$ is an eigenfunction of D . What are the corresponding eigenvalues? 3p
 c) Show that

$$\langle p, q \rangle = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^2} dx, \quad p, q \in \mathcal{P}_{\mathbb{R}},$$

is an inner product on $\mathcal{P}_{\mathbb{R}}$. *Hint:* A proof that these integrals exist is not required. Moreover, you may use without proof that if h is a continuous function such that $\int_{-\infty}^{\infty} h(x) dx = 0$ and $h \geq 0$, then $h(x) = 0$ for all $x \in \mathbb{R}$. Make clear where this statement is needed. 3p

d) Determine $\langle Dp_1, p_0 \rangle$ with p_0 and p_1 as in b). What do you conclude? *Hint:* Using the result of b) can simplify the calculation. 2p

Problem 3.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the 2-periodic function given by

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \in [-1, 0], \\ -3x + 1 & \text{if } x \in (0, 1), \end{cases} \quad x \in [-1, 1].$$

a) Visualize the graph of the function f by drawing at least two periods. 1p
 b) Determine the Fourier coefficients \hat{f}_k for all $k \in \mathbb{Z}$. *Hint:* Treat the cases $k = 0$ and $k \neq 0$ separately. 4p
 c) Argue why f can be expressed as a converging Fourier cosine series, i.e. $f(x) = \sum_{k=0}^{\infty} a_k \cos(k\pi x)$ for $x \in \mathbb{R}$ with $a_k \in \mathbb{R}$, and determine the coefficients a_k . 3p
 d) Use the result of c) to find the value of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \quad 2p$$

Problem 4.

Consider the initial value problem

$$t \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \quad (1)$$

$$u(x, 0) = g(x) \quad \text{for } x \in \mathbb{R}, \quad (2)$$

for $u = u(x, t) : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a given continuously differentiable and absolutely integrable function.

a) Apply Fourier transformation with respect to the x -variable to (1) and (2), and formulate the resulting initial value problems for $\hat{u}(s, \cdot)$ with $s \in \mathbb{R}$. 3p
 b) Show that for every $s \in \mathbb{R}$,

$$\hat{u}(s, t) = \hat{g}(s)e^{-i\frac{t^2}{2}s}, \quad t \geq 0,$$

solves the problem derived in a). 3p

c) If $\int_{-\infty}^{\infty} g(x) dx = 0$, what can you say about $\int_{-\infty}^{\infty} u(x, t) dx$ for $t > 0$? *Hint:* Use the result from b) with $s = 0$. 2p

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d)* For $\alpha > 0$ and $f : \mathbb{R} \rightarrow \mathbb{C}$ continuous and absolutely integrable, we define $f_\alpha : \mathbb{R} \rightarrow \mathbb{C}$ by $f_\alpha(x) = f(x - \alpha)$ for $x \in \mathbb{R}$. Show that

$$(\mathcal{F}f_\alpha)(s) = \hat{f}_\alpha(s) = e^{-ias}\hat{f}(s)$$

for all $s \in \mathbb{R}$.

3p

e) Use the results from b) and d) to find the solution $u(x, t) = g(x - \frac{t^2}{2})$ to (1) and (2).

3p

Problem 5.

Consider the function

$$f : \mathbb{R} \rightarrow \mathbb{C}, \quad t \mapsto \begin{cases} 1 - t^2 & \text{if } |t| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

as well as the continuous function $g : \mathbb{R} \rightarrow \mathbb{C}$ with

$$g(s) = \frac{\sin(s)}{s^3} - \frac{\cos(s)}{s^2} \quad \text{for } s \in \mathbb{R}, \quad s \neq 0.$$

a) Draw a picture of the graph of f .

1p

b) Determine the Fourier transform of f and show that it is a multiple of g , i.e. $\mathcal{F}f = \hat{f} = ag$ with $a \in \mathbb{C}$. What is the precise value of a ?

6p

c) Use Plancherel's formula and the result from b) to calculate $\int_{-\infty}^{\infty} |g(s)|^2 ds$.

3p

Problem 6*.

Let $k, m > 0$. The linear second order differential equation

$$mf'' = -kf \quad (3)$$

for $f = f(t) : \mathbb{R} \rightarrow \mathbb{C}$ describes a harmonic oscillator with m the mass of the particle and k the spring constant.

a) Rewrite (3) in terms of a vector-valued first order differential equation of the form

$$v' = Av \quad (4)$$

for $v = v(t) : \mathbb{R} \rightarrow \mathbb{C}^2$ with the matrix $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}$.

2p

b) Find the solution to (4) that satisfies the initial condition $v(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3p

c) Define $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $Hv = iAv$ for $v \in \mathbb{C}^2$ with A as in a). Show that H is a linear operator.

1p

d) Let \mathbb{C}^2 be endowed with the standard inner product. For which values of m and k is H Hermitian, i.e. $\langle Hv, w \rangle = \langle v, Hw \rangle$ for all $v, w \in \mathbb{C}^2$?

3p

e) Assume that $m = k$ and take v to be a solution to (4). Use d) to determine the expression $\frac{d}{dt} \|v(t)\|^2$ for all $t \in \mathbb{R}$. What is the geometric interpretation of these findings for the solution determined in b)?

3p

Sketch of suggested solutions

Please email errors and/or suggestions to c.kreisbeck@uu.nl.

Problem 1.

a) The length of v is $\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{1 + 1 + 0} = \sqrt{2}$ and similarly $\|w\| = \sqrt{2}$. Moreover,

$$\langle v, w \rangle = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 1.$$

b) We show that Z is non-empty and closed under addition and scalar multiplication:

- Choosing $a = b = 0$ shows that $0 \in Z$.
- Let $u, z \in Z$, then $z = av + bw$ and $u = cv + dw$ for $a, b, c, d \in \mathbb{R}$ and

$$z + u = av + bw + cv + dw = (a + c)v + (b + d)w.$$

Since $a + b, c + d \in \mathbb{R}$, we find that $z + u \in Z$.

- Let $z \in Z$ and $r \in \mathbb{R}$. Then $z = av + bw$ with $a, b \in \mathbb{R}$ and $rz = r(av + bw) = (ra)v + (rb)w$. As $ra, rb \in \mathbb{R}$, it follows that $rz \in Z$.

c) Following the Gram-Schmidt orthonormalization procedure, we set

$$v^* = \frac{v}{\|v\|} \stackrel{a)}{=} \frac{1}{\sqrt{2}}v,$$

and

$$\tilde{w} = w - \langle w, v^* \rangle v^* = w - \frac{\langle w, v \rangle}{\|v\|^2} v = w - \frac{1}{2}v = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

One can easily double-check that v^* and \tilde{w} are indeed orthogonal. It remains to normalize \tilde{w} . As

$$\|\tilde{w}\|^2 = \tilde{w}_1^2 + \tilde{w}_2^2 + \tilde{w}_3^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2 = \frac{3}{2},$$

we set

$$w^* = \frac{\tilde{w}}{\|\tilde{w}\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

By construction, the vectors v^* and w^* form an orthonormal basis of Z .

d) Let $z = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$. One obtains for the projection $\mathcal{P}z$ of z onto Z that

$$\mathcal{P}z = \langle z, v^* \rangle v^* + \langle z, w^* \rangle w^* = \frac{1}{2} \langle z, v \rangle v + \frac{4}{6} \langle z, \tilde{w} \rangle \tilde{w} = 3v + 2\tilde{w} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}.$$

Problem 2.

a) Let $p, q \in \mathcal{P}_{\mathbb{R}}$ and $r, s \in \mathbb{R}$, then by the sum rule of differentiation,

$$\begin{aligned}(D(rp + sq))(x) &= x^2(rp + sq)''(x) + 2x(rp + sq)'(x) = x^2(rp''(x) + sq''(x)) + 2x(rp'(x) + sq'(x)) \\ &= r(x^2p''(x) + 2xp'(x)) + s(x^2q''(x) + 2xq'(x)) = r(Dp)(x) + s(Dq)(x)\end{aligned}$$

for all $x \in \mathbb{R}$. This shows that D is a linear operator.

b) Let $k \in \mathbb{N}_0$. Then for $x \in \mathbb{R}$,

$$(Dp_k)(x) = x^2k(k-1)x^{k-2} + 2xkx^{k-1} = k(k-1)x^k + 2kx^k = (k^2 - k + 2k)x^k = (k^2 + k)p_k(x).$$

Hence, $Dp_k = (k^2 + k)p_k$. Since p_k is non-trivial, this shows that p_k is an eigenfunction of D and the corresponding eigenvalue is $k^2 + k$.

c) To show that $\langle \cdot, \cdot \rangle$ is an inner product, we prove linearity in the first argument, symmetry, positivity and definiteness:

- Linearity in the first argument and symmetry are immediate consequences of the linearity of the integral and the properties of \mathbb{R} . More precisely, for $p, \tilde{p}, q \in \mathcal{P}_{\mathbb{R}}$ and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\langle \alpha p + \beta \tilde{p}, q \rangle &= \int_{-\infty}^{\infty} (\alpha p(x) + \beta \tilde{p}(x))q(x)e^{-x^2} dx \\ &= \alpha \int_{-\infty}^{\infty} p(x)q(x)e^{-x^2} dx + \beta \int_{-\infty}^{\infty} \tilde{p}(x)q(x)e^{-x^2} dx = \alpha \langle p, q \rangle + \beta \langle \tilde{p}, q \rangle.\end{aligned}$$

Similarly one can prove that $\langle p, q \rangle = \langle q, p \rangle$.

- For all $p \in \mathcal{P}_{\mathbb{R}}$ it holds that $\langle p, p \rangle = \int_{-\infty}^{\infty} p(x)^2 e^{-x^2} dx \geq 0$, since the integrand $p(x)^2 e^{-x^2}$ is non-negative for every $x \in \mathbb{R}$.
- If $0 = \langle p, p \rangle = \int_{-\infty}^{\infty} p(x)^2 e^{-x^2} dx$, then the hint tells us that $p(x)^2 e^{-x^2} = 0$ for all $x \in \mathbb{R}$. Since $e^{-x^2} > 0$, it follows that $p(x)^2 = 0$ and thus $p(x) = 0$ for all $x \in \mathbb{R}$, meaning that p is the null polynomial.

d) Accounting for b) gives that $Dp_1 = 2p_1$. Then

$$\langle Dp_1, p_0 \rangle = 2\langle p_1, p_0 \rangle = \int_{-\infty}^{\infty} 2x p_1(x) p_0(x) e^{-x^2} dx = \lim_{N \rightarrow -\infty} e^{-N^2} - \lim_{N \rightarrow \infty} e^{-N^2} = 0.$$

This calculation shows that Dp_1 and p_0 are orthogonal.

Problem 3.

a) The visualization of f is left to the reader. We observe that f is piecewise continuously differentiable.

b) For $k = 0$ we have that

$$\hat{f}_0 = \frac{1}{2} \int_{-1}^1 f(x) dx = \int_0^1 -3x + 1 dx = -\frac{3}{2} + 1 = -\frac{1}{2}.$$

In the case $k \neq 0$, we use integration by parts to obtain

$$\begin{aligned}
\hat{f}_k &= \frac{1}{2} \int_{-1}^1 f(x) e^{-ik\pi x} dx = \underbrace{\frac{1}{2} \int_{-1}^1 e^{-ik\pi x} dx}_{=0} + \frac{3}{2} \left[\int_{-1}^0 x e^{-ik\pi x} dx - \int_0^1 x e^{-ik\pi x} dx \right] \\
&= \frac{3}{2} \left[\int_{-1}^0 x \cos(k\pi x) dx - \int_0^1 x \cos(k\pi x) dx - \underbrace{\int_{-1}^0 x \sin(k\pi x) dx + \int_0^1 x \sin(k\pi x) dx}_{=0} \right] \\
&= 3 \int_{-1}^0 x \cos(k\pi x) dx = -3 \int_{-1}^0 \frac{1}{k\pi} \sin(k\pi x) dx = \frac{3}{k^2\pi^2} (1 - \cos(k\pi)) \\
&= \begin{cases} 0 & \text{if } k \text{ even,} \\ \frac{6}{k^2\pi^2} & \text{if } k \text{ odd.} \end{cases}
\end{aligned}$$

c) Since f is 2-periodic, piecewise continuously differentiable and continuous, the Fourier inversion formula tells us that for $x \in \mathbb{R}$,

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\pi x} dx,$$

where the series on the right-hand side converges. As a consequence of the Euler formula we can write

$$\begin{aligned}
f(x) &= \hat{f}_0 + \sum_{k=1}^{\infty} (\hat{f}_k + \hat{f}_{-k}) \cos(k\pi x) + i(\hat{f}_k - \hat{f}_{-k}) \sin(k\pi x) \\
&= a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x), \quad x \in \mathbb{R},
\end{aligned}$$

with $a_0 = \hat{f}_0$, $a_k = \hat{f}_k + \hat{f}_{-k}$ and $b_k = i(\hat{f}_k - \hat{f}_{-k})$ for $k \in \mathbb{N}$. Since f is even, we have that $\hat{f}_k = \hat{f}_{-k}$ for all $k \in \mathbb{Z}$. Consequently, $b_k = 0$ and $a_k = 2\hat{f}_k$ for $k \in \mathbb{N}$, so that

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x) = \sum_{k=0}^{\infty} a_k \cos(k\pi x), \quad x \in \mathbb{R}. \quad (5)$$

Using the calculations in b), we have that $a_0 = -\frac{1}{2}$, and $a_k = 0$ for k even and $a_k = \frac{12}{k^2\pi^2}$ for k odd.

d) Setting $x = 0$ in (5) results in

$$1 = f(0) = -\frac{1}{2} + \sum_{k=1, k \text{ odd}} \frac{12}{k^2\pi^2},$$

or equivalently,

$$\frac{\pi^2}{8} = \sum_{k=1, k \text{ odd}} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Problem 4.

a) Observe that with \mathcal{F} denoting the Fourier transformation with respect to the x -variable, we obtain for every $t > 0$ that

$$\mathcal{F}(t \frac{\partial}{\partial x} u(\cdot, t))(s) = t(is) \hat{u}(s, t) = ist \hat{u}(s, t), \quad s \in \mathbb{R},$$

and

$$\mathcal{F}(\frac{\partial}{\partial t} u(\cdot, t))(s) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x, t) e^{-isx} dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x, t) e^{-isx} dx = \frac{\partial}{\partial t} \hat{u}(s, t), \quad s \in \mathbb{R}.$$

Hence, applying \mathcal{F} to (1) and (2) results in

$$\frac{\partial}{\partial t} \hat{u}(s, t) + ist \hat{u}(s, t) = 0 \quad \text{and} \quad \hat{u}(s, 0) = \hat{g}(s)$$

for every $s \in \mathbb{R}$, or in other words, $\hat{u}(s, \cdot)$ solves the first order differential equation with non-constant coefficients

$$v'(t) + ist v(t) = 0, \quad t > 0, \quad (6)$$

subject to the initial condition $v(0) = \hat{g}(s)$.

b) We verify that the suggested formula solves (6) by plugging it in. Indeed, for every $s \in \mathbb{R}$ we find with the chain rule that $\frac{d}{dt} e^{-i\frac{t^2}{2}s} = -ist e^{-i\frac{t^2}{2}s}$, so that

$$\frac{d}{dt} (\hat{g}(s) e^{-i\frac{t^2}{2}s}) + ist \hat{g}(s) e^{-i\frac{t^2}{2}s} = 0.$$

Moreover, $\hat{u}(s, 0) = \hat{g}(s) e^{-i\frac{0^2}{2}\cdot s} = \hat{g}(s)$. This shows that $\hat{u}(s, \cdot)$ solves the initial value problem derived in b).

c) Since $\hat{g}(0) = \int_{-\infty}^{\infty} g(x) dx$ and $\hat{u}(0, t) = \int_{-\infty}^{\infty} u(x, t) dx$ for every $t > 0$, we find in view of b) that

$$\int_{-\infty}^{\infty} u(x, t) dx = \hat{u}(0, t) = \hat{g}(0) e^{-i\frac{t^2}{2}\cdot 0} = \hat{g}(0) = \int_{-\infty}^{\infty} g(x) dx.$$

Consequently, if $\int_{-\infty}^{\infty} g(x) dx = 0$, then $\int_{-\infty}^{\infty} u(x, t) dx = 0$ for all $t > 0$.

d) It follows from the change of variables $y = x - \alpha$ that

$$\begin{aligned} \hat{f}_{\alpha}(s) &= \int_{-\infty}^{\infty} f_{\alpha}(x) e^{-isx} dx = \int_{-\infty}^{\infty} f(x - \alpha) e^{-isx} dx = \int_{-\infty}^{\infty} f(y) e^{-is(y+\alpha)} dy \\ &= e^{-is\alpha} \int_{-\infty}^{\infty} f(y) e^{-isy} dy = e^{-is\alpha} \hat{f}(s), \quad s \in \mathbb{R}. \end{aligned}$$

e) In view of b) and d) one obtains that

$$\hat{u}(s, t) = \hat{g}(s) e^{-i\frac{t^2}{2}s} = \hat{g}_{\frac{t^2}{2}}(s), \quad s \in \mathbb{R}, \quad t \geq 0.$$

Using Fourier inversion we infer for every $t \geq 0$ that

$$u(x, t) = \mathcal{F}^{-1}(\hat{u}(\cdot, t))(x) = g_{\frac{t^2}{2}}(x) = g\left(x - \frac{t^2}{2}\right), \quad x \in \mathbb{R}.$$

Problem 5.

a) The visualization of f is left to the reader. We observe that f is piecewise continuously differentiable and has compact support, in particular f is absolutely integrable.

b) If $s \neq 0$ we find that

$$\begin{aligned}\hat{f}(s) &= \int_{-\infty}^{\infty} f(t)e^{-ist} dt = \int_{-1}^1 e^{-ist} dt - \int_{-1}^1 t^2 e^{-ist} dt \\ &= \frac{2 \sin(s)}{s} + \frac{4 \sin(s)}{s^3} - \frac{4 \cos(s)}{s^2} - \frac{2 \sin(s)}{s} = 4g(s).\end{aligned}$$

In the third equality we have used the result of the following computation, for which we apply integration by parts twice,

$$\begin{aligned}\int_{-1}^1 t^2 e^{-its} dt &= \frac{2}{is} \int_{-1}^1 t e^{-ist} dt + \frac{1}{is} (e^{is} - e^{-is}) \\ &= \frac{2}{(is)^2} \int_{-1}^1 e^{-ist} dt - \frac{2}{(is)^2} (e^{is} + e^{-is}) + 2 \frac{\sin(s)}{s} \\ &= \frac{2}{-s^2} \left(\frac{1}{-is} e^{-is} + \frac{1}{is} e^{is} \right) + \frac{4 \cos(s)}{s^2} + \frac{2 \sin(s)}{s} = -\frac{4 \sin(s)}{s^3} + \frac{4 \cos(s)}{s^2} + \frac{2 \sin(s)}{s}.\end{aligned}$$

In the case $s = 0$ one has that

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(t) dt = 2 \int_0^1 1 - t^2 dt = 2 \left(1 - \frac{1}{3} \right) = \frac{4}{3}.$$

Finally, since \hat{f} is continuous, we see that $\hat{f} = ag$ with $a = 4$.

c) From b) and Plancherel's formula we infer that

$$\begin{aligned}\int_{-\infty}^{\infty} |g(s)|^2 ds &= \frac{1}{4^2} \int_{-\infty}^{\infty} |\hat{f}(s)|^2 ds = \frac{\pi}{8} \int_{-\infty}^{\infty} |f(s)|^2 ds = \frac{\pi}{8} \int_{-1}^1 (1 - t^2)^2 dt \\ &= \frac{\pi}{4} \int_0^1 1 - 2t^2 + t^4 dt = \frac{2\pi}{15}.\end{aligned}$$

Problem 6*.

a) Let $v(t) = \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}$ for all $t \in \mathbb{R}$. Then

$$v' = \begin{pmatrix} f' \\ f'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \begin{pmatrix} f \\ f' \end{pmatrix} = Av \quad \text{with } A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}.$$

b) Solving (4) with initial value $v(0) = e_1$ is equivalent to solving (3) subject to the initial conditions $f(0) = 1$ and $f'(0) = 0$. Since the roots of the polynomial $\lambda^2 + \frac{k}{m}$ are $\lambda_{1/2} = \pm i\sqrt{\frac{k}{m}}$, the general solution to (3) is $f(t) = \alpha e^{i\sqrt{\frac{k}{m}}t} + \beta e^{-i\sqrt{\frac{k}{m}}t}$ with $\alpha, \beta \in \mathbb{C}$. We infer from the initial conditions that

$$1 = f(0) = \alpha + \beta \quad \text{and} \quad 0 = f'(0) = i\sqrt{\frac{k}{m}}(\alpha - \beta).$$

Hence $\alpha = \beta = \frac{1}{2}$ and $f(t) = \cos(\sqrt{\frac{k}{m}}t)$ for $t \in \mathbb{R}$. Consequently, the sought solution is

$$v(t) = \begin{pmatrix} \cos(\sqrt{\frac{k}{m}}t) \\ -\sqrt{\frac{k}{m}} \sin(\sqrt{\frac{k}{m}}t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

c) For all $v, w \in \mathbb{C}^2$ and $\alpha, \beta \in \mathbb{C}$ one obtains that

$$H(\alpha v + \beta w) = iA(\alpha v + \beta w) = iA(\alpha v) + iA(\beta w) = \alpha iAv + \beta iAw = \alpha H(v) + \beta H(w),$$

which shows that H is a linear operator.

d) We will prove that H is Hermitian if and only if $k = m$. Observing that $Av = \begin{pmatrix} v_2 \\ -\frac{k}{m}v_1 \end{pmatrix}$ for $v \in \mathbb{C}^2$, we obtain for all $v, w \in \mathbb{C}^2$ that

$$\begin{aligned} \langle Hv, w \rangle - \langle v, Hw \rangle &= i\langle Av, w \rangle - (-i)\langle v, Aw \rangle = i(\langle Av, w \rangle + \langle v, Aw \rangle) \\ &= i(v_2\bar{w}_1 - \frac{k}{m}v_1\bar{w}_2 + v_1\bar{w}_2 - \frac{k}{m}v_2\bar{w}_1) = i(1 - \frac{k}{m})(v_1\bar{w}_2 + v_2\bar{w}_1). \end{aligned}$$

Since in general $v_1\bar{w}_2 + v_2\bar{w}_1 \neq 0$ (take e.g. $v = e_1$ and $w = e_2$), this implies that H is Hermitian if and only if $1 - \frac{k}{m} = 0$, or equivalently $k = m$.

e) Considering that v is a solution to (4) we find that

$$\begin{aligned} \frac{d}{dt} \|v(t)\|^2 &= \frac{d}{dt} \langle v(t), v(t) \rangle = \langle v'(t), v(t) \rangle + \langle v(t), v'(t) \rangle = \langle Av(t), v(t) \rangle + \langle v(t), Av(t) \rangle = \\ &= \langle -iHv(t), v(t) \rangle + \langle v(t), -iHv(t) \rangle = -i\langle Hv(t), v(t) \rangle + i\langle v(t), Hv(t) \rangle = 0, \quad t \in \mathbb{R}. \end{aligned}$$

In the last step we used the fact that H is Hermitian.

As a consequence, the norm of $v(t)$ is constant for all $t \in \mathbb{R}$. In the case $k = m$ the solution from b) lives on a circle in the two-dimensional real plane.